5. Continuum Dynamics

Present your solutions to the following problems using latex, if you have figures make sure they are publication quality, include your code in the solutions. Print your pdf files and bring them to class.

1. Real Capacitor.

Consider a capacitor in a box. Let all surfaces continue forever in the z-direction, so that this is a 2-d problem. Choose dimensions so that the plates are at $x = \pm 1/4$ and extend from $y = -1/4$ to $y = +1/4$. Place the left plate at $V = -1$ and the right at $V = 1$. Place the capacitor in a box from $x = -1..1$ and $y = -1..1$ held at $V = 0$. Work on a $(2N + 1) \times (2N + 1)$ grid.

(i) Determine $V(x, y)$ using the Jacobi method on a 201×201 grid. Make a contour plot of your result. Implement and describe a reasonable termination criterion.

(ii) Plot the error estimate in V vs. iteration for $N = 20, 40, 80$

(iii) Implement the Gauss-Seidel scheme and make the same plot as (ii).

(iv) Implement checkerboard SOR and make the same plot as (ii).

(v) Confirm that SOR is an $O(L^3)$ algorithm, while Jacobi is $O(L^4)$.

2. Advection-diffusion

(i) Obtain the analytic solution to

$$\partial_i D_{ij} \partial_j u - \vec{v} \cdot \nabla u = \frac{\partial u}{\partial t}$$

assuming a constant diffusivity tensor and velocity. You can assume a point contaminant source

$$u(\vec{r}, t = 0) = \xi_0 \delta(\vec{r})$$

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Think carefully about how ξ_0 relates to u.

(ii) Solve the (1+1)-d case using the Crank-Nicolson method. You can take $v = 1$ and $D = 1$ for convenience (real values are closer to $v = 0.1$ m/day and $D = 8 \cdot 10^{-5}$ m^2/day). A simple algorithm for inverting a tridiagonal matrix is given on the course web site. Plot u/ξ_0 vs x for a variety of times (on one graph).

(iii) Consider the ADI method applied to the (2+1)-d version of this problem. (a) Obtain the stability condition from von Neumann analysis. (b) Consider the x-step portion of the ADI method. How would you obtain $u_{ij}^{n+1/2}$? Specifically, is it better to map the matrix to a vector $u_{ij} \rightarrow u_{I(i,j)}$ or to solve the tridiagonal problem many times (in j)?