Phys 3707, Assignment 8 – Spin Systems

1. Lanczos Algorithm

The Lanczos algorithm is a method to solve large quantum problems that generates an orthogonal basis using a recursion relation

$$|\phi_{n+1}\rangle = H|\phi_n\rangle + a_n|\phi_n\rangle + b_n^2|\phi_{n-1}\rangle$$

with

$$a_n = \frac{\langle\phi_n|H|\phi_n\rangle}{\langle\phi_n|\phi_n\rangle}$$

and

$$b_n^2 = \frac{\langle\phi_n|\phi_n\rangle}{\langle\phi_{n-1}|\phi_{n-1}\rangle}.$$

The benefit of this basis is that the Hamiltonian is tri-diagonal, which permits easy diagonalization.

Prove that the set of generated Lanczos vectors are mutually orthogonal.

2. Classical $O(n)$ Spin Model.

(i) Use the Hubbard-Stratanovich transformation and mean field theory to obtain the partition function for the classical $O(n)$ spin model.

(ii) Evaluate the n-dimensional integral that arises in Z to $O(\phi^4)$.

(iii) Hence obtain expressions for A and B for

$$\Gamma(\vec{m}_i) = -\frac{1}{2}\sum_{ij}\vec{m}_i J_{ij}\vec{m}_j + \frac{A}{2}T\sum_i \vec{m}_i^2 + \frac{B}{4}\sum_i(\vec{m}_i^2)^2.$$

(iv) Obtain expressions for the longitudinal and transverse spin-spin correlation functions for T below, at, and above the critical temperature.

3. Staggered Magnetization

The staggered magnetization operator is defined by

$$N_z = \frac{1}{V}\sum_i(-)^i\sigma_i^z.$$

(i) Argue that $\langle N_z\rangle = 0$ for a general antiferromagnet.

(ii) Compute $\langle N_z^2\rangle$ to $O(g^2)$ in the ground state of the 2d square lattice nearest neighbour anisotropic Heisenberg antiferromagnet.

4. The Hubbard Model

Consider the two-site Hubbard model at half filling with periodic boundary conditions.

(i) Carefully examine momentum eigenstates and argue that two states contribute to the $S^z_{\text{tot}} = 0$ sector.

(ii) Obtain an explicit expression for H in this basis and compute the eigenvalues and eigenvectors.