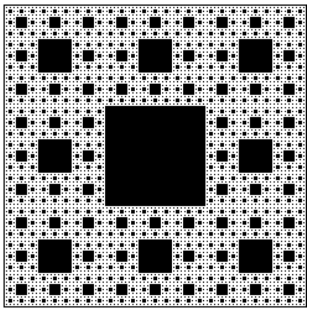
Phys 3707, Assignment 9 – The Renormalization Group

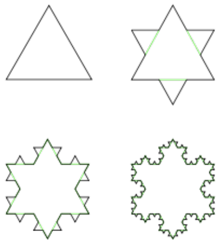
1. Fractals.

The fractal dimension d is defined by $v = r^d$ where v is the volume of an object and r is a linear dimension for the object.

(a) Compute the fractal dimension of the Sierpinksi carpet.

(b) Compute the fractal dimenion of the Koch snowflake.





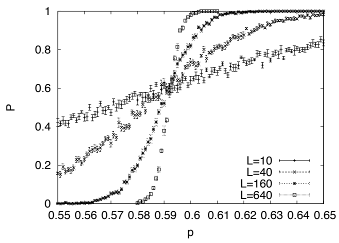
2. RG and Percolation.

Consider an $L \times L$ checkerboard of squares. Colour each square with probability p. We define a cluster to be a collection of coloured squares that touch (ie, are neighbors). A cluster is said to percolate if it connects the top and bottom of the grid. Other definitions of percolation are possible and yield simiar behaviour (universality!).

The first question we address is what is the probability of percolation as a function of the colouring probability p and the grid size, L? Call this $P(p, L)$. It is possible to analytically compute the percolation probability for small grids, but the problem rapidly becomes intractable since the number of configuration grows as 2^{L^2}.

(a) Show that $P(p, 2) = p^2(2 - p^2)$.

A numerical determination of P yields results like shown below.



The percolation probability rapidly gets steeper as L gets larger. In fact as the lattice size approaches infinity the percolation probability approaches a step function:

$$\lim_{L \to \infty} P(p, L) = \theta(p - p_c)$$

where p_c is the critical probability. In the bulk limit generating configurations with a density below p_c never yields a percolating cluster, while it always yields one above p_c. While reminiscent of a phase transition, this remarkable property of percolation has nothing to do with dynamics, it is purely statistical in nature!

At p_c percolating cluster are self-similar, which means they are invariant under coarse graining. In this case it is convenient to implement coarse graining by replacing a sublattice of size $b \times b$ with a single superlattice site. The superlattice site will be deemed to be occupied if the original sublattice percolated. Thus an $L \times L$ lattice is mapped to an $L/b \times L/b$ superlattice with an effective percolation probability called p'. It is clear that (i) the superlattice is simply another percolation problem with a different scale ξ' and probability p' (ii) the coarse graining process can be repeated yielding a real space renormalization map.

The effective correlation length is related to the original correlation length in a specific way:

$$\xi = k(p - p_c)^{-\nu} \tag{1}$$
$$\xi' = kb(p' - p_c)^{-\nu}. \tag{2}$$

At criticality one must have $p = p'$ and $\xi = \xi'$ thus

$$\nu = \lim_{p' \to p} \frac{\log b}{\log\left(\frac{p'-p_c}{p-p_c}\right)}. \tag{3}$$

(b) Implement the real space renormalization group on a triangular lattice and estimate the the critical probability and the critical exponent ν. Compare your answer to the exact results of $p_c = 1/2$ and $\nu = 4/3$.