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Radiative Corrections as the Origin of Spontaneous Symmetry Breaking*

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We investigate the possibility that radiative corrections may produce spontaneous symmetry breakdown in theories for which the semiclassical (tree) approximation does not indicate such breakdown. The simplest model in which this phenomenon occurs is the electrodynamics of massless scalar mesons. We find (for small coupling constants) that this theory more closely resembles the theory with an imaginary mass (the Abelian Higgs model) than one with a positive mass; spontaneous symmetry breaking occurs, and the theory becomes a theory of a massive vector meson and a massive scalar meson. The scalar-to-vector mass ratio is computable as a power series in e, the electromagnetic coupling constant. We find, to lowest order, $m^2(S)/m^2(V) = (3/2\pi)(e^2/4\pi)$. We extend our analysis to non-Abelian gauge theories, and find qualitatively similar results. Our methods are also applicable to theories in which the tree approximation indicates the occurrence of spontaneous symmetry breakdown, but does not give complete information about its character. (This typically occurs when the scalar-meson part of the Lagrangian admits a greater symmetry group than the total Lagrangian.) We indicate how to use our methods in these cases.

I. INTRODUCTION

Massless scalar electrodynamics, the theory of the electromagnetic interactions of a mass-zero charged scalar field, has had a bad name for a long time now; the attempt to interpret this theory consistently has led to endless paradoxes.[1] In this paper we describe how nature avoids these paradoxes: Massless scalar electrodynamics does not remain massless, nor does it remain electrodynamics; both the scalar meson and the photon acquire a mass as a result of radiative corrections.

The preceding statement may appear less oracular if we imbed massless scalar electrodynamics in a larger family of theories with a mass. If we write the complex scalar fields in terms of two real fields, φ_1 and φ_2, the Lagrange density is[2]

$$\mathcal{L} = -\tfrac{1}{4}(F_{\mu\nu})^2 + \tfrac{1}{2}(\partial_\mu\varphi_1 - eA_\mu\varphi_2)^2 + \tfrac{1}{2}(\partial_\mu\varphi_2 + eA_\mu\varphi_1)^2 - \tfrac{1}{2}\mu^2({\varphi_1}^2 + {\varphi_2}^2) - \frac{\lambda}{4!}({\varphi_1}^2 + {\varphi_2}^2)^2 , \qquad (1.1)$$

plus renormalization counterterms. (The quartic self-coupling is required for renormalizability, to cancel the logarithmic divergence that arises in the amplitude for scalar Coulomb scattering.) It is widely believed that if μ^2 is positive, this is a normal field theory.[3] The particle spectrum consists of a charged scalar particle, its antiparticle, and a massless photon. On the other hand, if μ^2 is negative, the theory is unstable; spontaneous symmetry breaking takes place.[4] The scalar field acquires a vacuum expectation value, the Higgs phenomenon occurs, and at the end we are left with a massive neutral scalar meson and a massive neutral vector meson.[5]

Our assertion is that the massless theory is like the second case rather than the first; the only difference is that the driving mechanism of the instability is not a negative mass term in the Lagrangian, but certain effects of higher-order processes involving virtual photons.

Other interesting things happen in this theory. The Lagrange density (1.1), with vanishing mass, involves two free parameters, e and λ. After spontaneous breaking, the theory is still expressed in terms of two parameters; these are most conveniently chosen to be e and the vacuum expectation of the field, $\langle\varphi\rangle$. The surprising thing is that we have traded a dimensionless parameter, λ, on which physical quantities can depend in a complicated way, for a dimensional one, $\langle\varphi\rangle$, on which physical quantities must depend in a trivial way, governed by dimensional analysis. We call this phenomenon dimensional transmutation, and argue that it is a general feature of spontaneous symmetry breaking in fully massless field theories.

One consequence of dimensional transmutation is that all dimensionless quantities must be functions of e alone. In particular we find that the scalar-to-vector mass ratio is given by

$$\frac{m^2(S)}{m^2(V)}=\frac{3}{2\pi}\frac{e^2}{4\pi}\,, \tag{1.2}$$

plus higher-order corrections. We think that this is the most interesting result of our analysis; we believe it is the first time a small mass ratio (rather than a small mass difference) has been found as a natural consequence of the presence of a small dimensionless coupling constant.

The outline of our paper is as follows: Section II is a brief review of the formalism upon which our work is based, the method of generating functionals and effective potentials. Section III is a detailed sample computation in the simplest case, massless quartically self-interacting meson theory. Here we show how to treat in our formalism the infrared divergences which occur in massless field theories. Section IV contains the computations for massless scalar electrodynamics, and a discussion of their reliability. Section V is an explanation of how to apply the renormalization group of Gell-Mann and Low to our problem; we use this method to extend the domain of reliability of our earlier results. The reader who just wants to get a general idea of what we are up to is strongly advised to skip Sec. V, which is independent of nearly everything else in the paper. Section VI extends our methods to theories of multiplets of massless scalar mesons interacting with massless non-Abelian gauge fields and massless fermions. We obtain closed forms for the lowest-order effects of virtual gauge particles in the general case; these forms are valid even in theories in which there is a negative-mass term driving symmetry breakdown, and therefore may be of use to readers who do not share our fascination with fully massless theories. Section VII lists our conclusions and makes some speculations.

Two appendixes deal with matters peripheral to our main line of argument. One shows that certain interesting properties of our explicit lowest-order computations persist in higher orders; the other discusses the sense in which the massless theories we study are the limits of massive theories as the mass goes to zero.

II. FUNCTIONAL METHODS AND THE EFFECTIVE POTENTIAL

In this section we review the functional methods introduced into quantum field theory by Schwinger, and extended to the study of spontaneous symmetry breakdown by Jona-Lasinio.[6] It is common to study spontaneous symmetry breakdown in the so-called semiclassical approximation, that is to say, to search for minima of the potential, the negative sum of all the nonderivative terms in the Lagrange density. The method of Jona-Lasinio enables us to define a function, called the effective potential, such that the minima of the effective potential give, without any approximation, the true vacuum states of the theory. Furthermore, there exists a diagrammatic expansion for the effective potential, such that the first term in this expansion reproduces the semiclassical approximation. From our viewpoint, the virtue of this method is that it enables us to compute higher-order corrections while still retaining a great advantage of the semiclassical approximation: the ability to survey all possible vacua simultaneously. Thus we are able to investigate the character of the unstable but symmetric theory described by the same Lagrangian which governs a given Goldstone-type asymmetric theory, even in the presence of renormalization effects. In other formalisms for computing higher-order corrections to spontaneously broken symmetries, such investigations are much more difficult. In addition, we are able to investigate cases in which the radiative corrections qualitatively change the structure of the theory (e.g., by turning minima in the effective potential into maxima); in other formalisms, it is much more difficult to detect the occurrence of such phenomena.

A. General Formalism and Definitions

For notational simplicity, we shall restrict ourselves in this section to the theory of a single scalar field, φ, whose dynamics are described by a Lagrange density, $\mathcal{L}(\varphi, \partial_\mu\varphi)$. The generalization to more complicated cases is trivial. Let us consider the effect of adding to the Lagrange density a linear coupling of φ to an external source, $J(x)$, a c-number function of space and time:

$$\mathcal{L}(\varphi, \partial_\mu\varphi) \to \mathcal{L} + J(x)\varphi(x) \ . \tag{2.1}$$

The connected generating functional, $W(J)$, is defined in terms of the transition amplitude from the vacuum state in the far past to the vacuum state in the far future, in the presence of the source $J(x)$,

$$e^{iW(J)} = \langle 0^+ | 0^- \rangle_J \ . \tag{2.2}$$

We can expand W in a functional Taylor series

$$W = \sum_n \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \, G^{(n)}(x_1 \cdots x_n) J(x_1) \cdots J(x_n) \ . \tag{2.3}$$

It is well-known that the successive coefficients in this series are the *connected Green's functions;* $G^{(n)}$ is the sum of all connected Feynman diagrams with n external lines.

The classical field, φ_c, is defined by

$$\varphi_c(x) = \frac{\delta W}{\delta J(x)} = \left[\frac{\langle 0^+ | \varphi(x) | 0^- \rangle}{\langle 0^+ | 0^- \rangle}\right]_J \ . \tag{2.4}$$

The *effective action* $\Gamma(\varphi_c)$, is defined by a functional Legendre transformation

$$\Gamma(\varphi_c) = W(J) - \int d^4x \, J(x)\varphi_c(x) \ . \tag{2.5}$$

From this definition, it follows directly that

$$\frac{\delta \Gamma}{\delta \varphi_c(x)} = -J(x) \ . \tag{2.6}$$

This equation will shortly turn out to be critical in the study of spontaneous breakdown of symmetry. The effective action may be expanded in a manner similar to that of (2.3):

$$\Gamma = \sum_n \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \, \Gamma^{(n)}(x_1 \cdots x_n) \varphi_c(x_1) \cdots \varphi_c(x_n) \ . \tag{2.7}$$

It is possible to show that the successive coefficients in this series are the *1PI Green's functions* (sometimes called proper vertices); $\Gamma^{(n)}$ is the sum of all 1PI Feynman diagrams with n external lines. [A 1PI (one-particle-irreducible) Feynman diagram is a connected diagram that cannot be disconnected by cutting a single internal line. By convention, 1PI diagrams are evaluated with no propagators on the external lines.] There is an alternative way to expand the effective action: Instead of expanding in powers of φ_c, we can expand in powers of momentum (about the point where all external momenta vanish). In position space, such an expansion looks like

$$\Gamma = \int d^4x \left[-V(\varphi_c) + \tfrac{1}{2}(\partial_\mu \varphi_c)^2 Z(\varphi_c) + \cdots \right] \ . \tag{2.8}$$

$V(\varphi_c)$ – an ordinary function, not a functional – is called the effective potential. By comparing the expansions (2.7) and (2.8), it is easy to see that the nth derivative of V is the sum of all 1PI graphs with n vanishing external momenta. In tree approximation (that is to say, neglecting all diagrams with closed loops), V is just the ordinary potential, the negative sum of all nonderivative terms in the Lagrange density.

The usual renormalization conditions of perturbation theory can be expressed in terms of the functions that occur in (2.8). For example, if we define the squared mass of the meson as the value of the inverse propagator at zero momentum, then

$$\mu^2 = \left.\frac{d^2V}{d\varphi_c^{\,2}}\right|_0 \ . \tag{2.9a}$$

Likewise, if we define the four-point function at zero external momenta to be the coupling constant, λ, then

$$\lambda = \left.\frac{d^4V}{d\varphi_c^{\,4}}\right|_0 \ . \tag{2.9b}$$

Similarly, the standard condition for the normalization of the field becomes

$$Z(0) = 1 \ . \tag{2.9c}$$

[The alert reader may wonder what happens to these conditions in massless field theories, for which the Green's functions have logarithmic singularities (infrared divergences) when the external momenta vanish. The answer is that they survive with only minor modifications, as we will show in Sec. III.]

We are now ready to apply this apparatus to the study of spontaneous symmetry breaking. Let us suppose our Lagrange density possesses an internal symmetry; for simplicity, let us imagine it to be the transformation $\varphi \to -\varphi$. Then, spontaneous symmetry breaking occurs if the quantum field φ develops a nonzero vacuum expectation value, even when the source $J(x)$ vanishes. From Eqs. (2.4) and (2.6), this occurs if

$$\frac{\delta \Gamma}{\delta \varphi_c} = 0 , \qquad (2.10)$$

for some nonzero value of φ_c. Further, since we are typically only interested in cases where the vacuum expectation value is translationally invariant (that is to say, we are not interested in the spontaneous breakdown of momentum conservation), we can simplify this to

$$\frac{dV}{d\varphi_c} = 0 , \qquad (2.11)$$

for some nonzero value of φ_c. The value of φ_c for which the minimum occurs, which we denote by $\langle \varphi \rangle$, is the expectation value of φ in the new (asymmetric) vacuum. It is easy to see that if this situation is to be stable under small external perturbations, the stationary point given by Eq. (2.11) must be a minimum of the effective potential.

To explore the properties of the spontaneously broken theory, we define a new quantum field with vanishing vacuum expectation value,

$$\varphi' = \varphi - \langle \varphi \rangle \ . \qquad (2.12a)$$

This generates a corresponding redefinition of the classical field,

$$\varphi_c' = \varphi_c - \langle \varphi \rangle , \qquad (2.12b)$$

from which it immediately follows that the actual mass, coupling constant, etc., are computable from equations exactly like the Eqs. (2.9), except that the derivatives are evaluated at $\langle \varphi \rangle$, rather than at zero.

B. The Loop Expansion

Knowledge of the effective potential is knowledge of the structure of spontaneous symmetry breakdown. Unfortunately, except for trivial models, we do not know the effective potential; to calculate it requires an infinite summation of Feynman diagrams, a task beyond our computational abilities. Thus, it is important to know a sensible approximation method for V. We shall now attempt to show that one such sensible method is the loop expansion: first summing all diagrams with no closed loops (tree graphs), then those with one closed loop, etc.[7] Of course, each stage in this expansion also involves an infinite summation, but, as we shall show in Sec. III, this summation is trivial.

Let us introduce a parameter a into our Lagrange density, by defining

$$\mathcal{L}(\varphi, \partial_\mu \varphi, a) \equiv a^{-1} \mathcal{L}(\varphi, \partial_\mu \varphi) \ . \qquad (2.13)$$

We shall now show that the loop expansion is equivalent to a power-series expansion in a. Let P be the power of a associated with any graph. Then it is easy to see that

$$P = I - V , \qquad (2.14)$$

where I is the number of internal lines in the graph and V is the number of vertices. This is because the propagator, being the inverse of the differential operator occuring in the quadratic terms in $\mathcal{L}$, carries a factor of a, while every vertex carries a factor of a^{-1}. (Note that it is important that we are dealing with 1PI graphs, for which there are no propagators attached to external lines.) On the other hand, the number of loops, L is given by

$$L = I - V + 1 \ . \qquad (2.15)$$

This is because the number of loops in a diagram is equal to the number of independent integration momenta; every internal line contributes one integration momentum, but every vertex contributes a δ function that reduces the number of independent momenta by one, except for one δ function that is left over for over-all energy-momentum conservation. Combining Eqs. (2.14) and (2.15), we find that

$$P = L - 1 , \qquad (2.16)$$

the desired result.

The point of this analysis is *not* that the loop expansion is a good approximation scheme because a is a small parameter; indeed, a is equal to one. (However, it is certainly no worse than ordinary perturbation theory for small coupling constants, since the set of graphs with n loops or less certainly includes, as a subset, all graphs of nth order or less in the coupling constants.) The point is, rather, since the loop expansion corresponds to expansion in a parameter that multiplies the *total* Lagrange density, it is unaffected by shifts of fields [such as in Eq. (2.12)], and by the redefinition of the division of the Lagrangian into free and interacting parts associated with such shifts. The n-loop approximation to the effective potential thus preserves what we have identified as the main advantage of the effective potential method; it enables us to survey all possible vacuum states at once, and to compute higher-order corrections before deciding which vacuum the theory finally picks.

III. A SAMPLE COMPUTATION AND THE PROBLEM OF INFRARED DIVERGENCES

In this section we shall present a detailed computation of the one-loop approximation to the effective potential in the simplest possible case, the theory of a massless, quartically self-interacting meson field. The Lagrange density for this theory is

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\varphi)^2 - \frac{\lambda}{4!}\varphi^4 + \tfrac{1}{2}A(\partial_\mu\varphi)^2 - \tfrac{1}{2}B\varphi^2 - \frac{1}{4!}C\varphi^4 , \tag{3.1}$$

where A, B, and C are the usual wave-function, mass, and coupling-constant renormalization counterterms to be determined self-consistently, order by order in the expansion, by imposing the definitions of the scale of the renormalized field, the renormalized mass, and the renormalized coupling constant. (Note that a mass renormalization term is present, even though we are studying the massless theory; this is because the theory posesses no symmetry that would guarantee vanishing bare mass in the limit of vanishing renormalized mass.[8])

To lowest order (tree approximation) only one graph contributes, shown in Fig. 1. Thus we obtain

$$V = \frac{\lambda}{4!}\varphi_c{}^4 . \tag{3.2a}$$

To next order (one-loop approximation), we have the infinite series of polygon graphs shown in Fig. 2, as well as the contributions from the mass and coupling-constant counterterms. Thus we obtain

$$V = \frac{\lambda}{4!}\varphi_c{}^4 - \tfrac{1}{2}B\varphi_c{}^2 - \frac{1}{4!}C\varphi_c{}^4 + i\int\frac{d^4k}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{\frac{1}{2}\lambda\varphi_c{}^2}{k^2+i\epsilon}\right)^n , \tag{3.2b}$$

where B and C are, as usual in renormalization theory, only to be evaluated to lowest order in our expansion parameter, in this case the loop-counting parameter a.

Certain numerical factors in this expression require further explanation:

(1) The over-all factor of i comes from the definition of W, Eq. (2.2).

(2) The factor of $\frac{1}{2}$ in the numerator of the fraction is a Bose statistics factor; interchange of the two external lines at the same vertex does not lead to a new graph, and therefore the $1/4!$ in the definition of the coupling is incompletely canceled.

(3) The $1/2n$ is a combinatoric factor; rotation or reflection of the n-sided polygon does not lead to a new contraction in the Wick expansion, and therefore the $1/n!$ in Dyson's formula is incompletely canceled.

At first glance, the expression (3.2) seems hideously infrared divergent; each term in the sum is worse than the one before. However, considerable improvement is obtained if we sum the series

$$V = \frac{\lambda}{4!}\varphi_c{}^4 - \tfrac{1}{2}B\varphi_c{}^2 - \frac{1}{4!}C\varphi_c{}^4 + \tfrac{1}{2}\int\frac{d^4k}{(2\pi)^4}\ln\left(1+\frac{\lambda\varphi_c{}^2}{2k^2}\right) , \tag{3.3}$$

where, in this expression, we have rotated the integral into Euclidean space and dropped the $i\epsilon$. As we see, the apparent infrared divergence has been turned into a logarithmic singularity at φ_c equals zero. This is reminiscent of the phenomenon we would have encountered had we attempted to compute the radiative corrections to the propagator in this theory. As is well known, these behave, to lowest nontrivial order, like $p^2\ln p^2$; had we been so foolish as to attempt to calculate this function by computing its power-series expansion at p^2 equals zero, we would have found a sequence of increasingly infrared-divergent terms, just as in Eq. (3.2). The two situations are precisely parallel; in momentum space, we can avoid the infrared divergences by staying away from vanishing momentum; here, even though all our momenta vanish, we can avoid them by staying away from vanishing φ_c.

[Equation (3.3) also has an apparent logarithmic singularity in the coupling constant. However, as we shall show immediately, this singularity is illusory; it is eaten by the renormalization counterterms.]

Of course, the integral in Eq. (3.3) is still ultra-

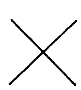

FIG. 1. The no-loop approximation for the effective potential.

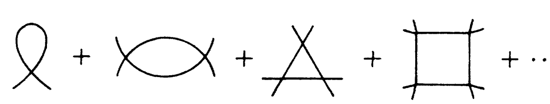

FIG. 2. The one-loop approximation for the effective potential.

violet-divergent; to evaluate it, we cut off the integral at $k^2=\Lambda^2$, and obtain

$$V=\frac{\lambda}{4!}\varphi_c^{\,4}+\tfrac{1}{2}B\varphi_c^{\,2}+\frac{1}{4!}C\varphi_c^{\,4}+\frac{\lambda\Lambda^2}{64\pi^2}\varphi_c^{\,2}+\frac{\lambda^2\varphi_c^{\,4}}{256\pi^2}\left(\ln\frac{\lambda\varphi_c^{\,2}}{2\Lambda^2}-\frac{1}{2}\right), \qquad (3.4)$$

where we have thrown away terms that vanish as Λ^2 goes to infinity. We can now determine the value of the renormalization counterterms by imposing the definitions of the renormalized mass and coupling constant.

We want the renormalized mass to vanish; following Eq. (2.9a), we find

$$\left.\frac{d^2V}{d\varphi_c^{\,2}}\right|_0=0\ . \qquad (3.5)$$

This implies that

$$B=-\frac{\lambda\Lambda^2}{32\pi^2}\ . \qquad (3.6)$$

Unfortunately, we cannot follow Eq. (2.9b) to define the renormalized coupling constant; the fourth derivative of V at the origin does not exist, because of the logarithmic infrared singularity. Once again, this is reminiscent of the situation that exists in momentum space for the same theory. There, the coupling constant can not be defined at the usual mass-shell symmetry point, because it is on top of the logarithmic singularity in momentum space; the standard strategy is to define the coupling constant at some off-mass-shell position in momentum space, away from the singularity. Here, we should lose much of the simplicity of our computation if we attempted to extend it away from vanishing momentum to do exactly the same thing; however, it is easy to do a parallel thing, and define the coupling constant at a point away from the singularity in classical-field space. That is to say, we define the renormalized coupling constant by

$$\left.\frac{d^4V}{d\varphi_c^{\,4}}\right|_M=\lambda\ , \qquad (3.7)$$

where M is some number with the dimensions of a mass. We emphasize that M is completely arbitrary, just as is the corresponding quantity in the momentum-space analysis; different choices for M will lead to different definitions of the coupling constant, different parametrizations of the theory, but any nonzero M is as good as any other. Although we do not need to know the wave-function renormalization counterterm, A, for the computation we are now doing, we remark that the standard condition for defining the scale of the field, Eq. (2.9c), is afflicted by the same infrared singularity as Eq. (2.9b). We avoid this difficulty as we avoided the previous one, by replacing Eq. (2.9c) by

$$Z(M)=1\ . \qquad (3.8)$$

Imposing Eq. (3.7), we find

$$C=-\frac{3\lambda^2}{32\pi^2}\left(\ln\frac{\lambda M^2}{2\Lambda^2}+\frac{11}{3}\right)\ . \qquad (3.9)$$

Putting all of this together, we find

$$V=\frac{\lambda}{4!}\varphi_c^{\,4}+\frac{\lambda^2\varphi_c^{\,4}}{256\pi^2}\left(\ln\frac{\varphi_c^{\,2}}{M^2}-\frac{25}{6}\right)\ . \qquad (3.10)$$

This is our final expression for the effective potential in the one-loop approximation and completes our sample computation.

Several comments are in order:

(1) This is a renormalizable theory; so we should expect all dependence on the cutoff to disappear from our final expression for V; this is indeed the case.

(2) As we remarked earlier, the violent infrared singularities in the individual diagrams have become a singularity at the origin of classical-field space. We show in Appendix A that this is true to all orders in the loop expansion.

(3) More surprisingly, but as promised, the logarithmic dependence on the coupling constant has also disappeared. In Appendix A we show that this is also true to higher orders; when all the dust of renormalization has settled, the n-loop contribution to V is simply proportional to λ^{n+1}.

(4) It is easy to check explicitly that the renormalization mass, M, is indeed an arbitrary parameter, with no effect on the physics of the problem. If we pick a different mass, M', then we define a new coupling constant

$$\lambda'=\left.\frac{d^4V}{d\varphi_c^{\,4}}\right|_{M'}=\lambda+\frac{3\lambda^2}{32\pi^2}\ln\frac{M'^2}{M^2}\ . \qquad (3.11)$$

Equation (3.10) may be rewritten as

$$V=\frac{\lambda'}{4!}\varphi_c^{\,4}+\frac{\lambda'^2\varphi_c^{\,4}}{256\pi^2}\left(\ln\frac{\varphi_c^{\,4}}{M'^2}-\frac{25}{6}\right)+O(\lambda^3)\ . \qquad (3.12)$$

This is simply a reparametrization of the same function, to the order in which we are working; it is a change of definitions, not a change of physics.

(The learned reader may remember that, in momentum space, where similar arbitrary renormalization masses enter for massless field theories, there exists a method of "improving" perturbation theory to obtain an expansion in which the independence of the renormalization masses is exact to every order. This method is the renormalization group of Gell-Mann and Low. It can be transferred bodily to the formalism we are using, and we will do so in Sec. V.)

(5) Since the logarithm of a small number is negative, it appears as though the one-loop corrections have turned the minimum at the origin into a maximum, and caused a new minimum to appear away from the origin – that is to say, that the one-loop corrections have generated spontaneous symmetry breaking. Alas, appearances are deceptive: The apparent new minimum occurs at a value of φ_c determined by

$$\lambda \ln\frac{\langle\varphi\rangle^2}{M^2} = -\frac{32}{3}\pi^2 + O(\lambda) \; . \tag{3.13}$$

Since we expect higher orders to bring in higher powers of $\lambda\ln(\varphi_c{}^2/M^2)$, the new minimum lies very far outside the expected range of validity of the one-loop approximation, even for an arbitrarily small coupling constant, and must be rejected as an artifact of our approximation. As we shall see shortly, though, there do exist physically interesting theories for which the effective potential has a very similar form, and to which the same criticism can not be applied. Among them is massless scalar electrodynamics.

IV. MASSLESS SCALAR ELECTRODYNAMICS

We shall now apply the apparatus developed in the last two sections to massless scalar electrodynamics, the theory of a massless charged meson minimally coupled to the electrodynamic field. We write the charged meson field in terms of two real fields, φ_1 and φ_2, and write the Lagrange density as

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}(F_{\mu\nu})^2 + \tfrac{1}{2}(\partial_\mu\varphi_1 - eA_\mu\varphi_2)^2 \\ &+ \tfrac{1}{2}(\partial_\mu\varphi_2 + eA_\mu\varphi_1)^2 - \frac{\lambda}{4!}(\varphi_1{}^2+\varphi_2{}^2)^2 \\ &+ \text{counterterms} \; ,\end{aligned} \tag{4.1}$$

where

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu \; . \tag{4.2}$$

The calculation of the effective potential can be somewhat simplified if we realize that it can only depend on

$$\varphi_c{}^2 = \varphi_{1c}{}^2 + \varphi_{2c}{}^2 \; . \tag{4.3}$$

Thus, we need only compute graphs with all the external lines φ_1's. Furthermore, if we work in Landau gauge, where the photon propagator is

$$D_{\mu\nu} = -i\,\frac{g_{\mu\nu} - k_\mu k_\nu/k^2}{k^2 + i\epsilon} \; , \tag{4.4}$$

then the contribution of any graph of the type shown in Fig. 3 vanishes. This is because the external momentum is zero; therefore the momentum of the internal meson is the same as that of the internal photon, and vanishes when it is contracted with the photon propagator.

Thus there are only three classes of graphs to compute: those of the type shown in Fig. 2, with a φ_1 running around the polygon, those of the same type, but with a φ_2 running around the polygon, and those shown in Fig. 4, with a photon running around the polygon. Aside from trivial numerical factors, all are of exactly the same structure as the graphs considered in Sec. III. After some straightforward computation, we obtain

$$V = \frac{\lambda}{4!}\,\varphi_c{}^4 + \left(\frac{5\lambda^2}{1152\pi^2} + \frac{3e^4}{64\pi^2}\right)\varphi_c{}^4\left(\ln\frac{\varphi_c{}^2}{M^2} - \frac{25}{6}\right) \; . \tag{4.5}$$

This function, like the one discussed in Sec. III, has a minimum away from the origin. Here, however, the minimum need not be illusory. In Sec. III, the minimum arose from balancing a term of order λ against a term of order $\lambda^2\ln(\varphi/M)$; thus, for small λ, it inevitably occurred at large $\ln(\varphi/M)$, outside the expected domain of validity of our approximation. Here, even for an arbitrarily small coupling constant, we can obtain a minimum by balancing a term of order λ against a term of order $e^4\ln(\varphi/M)$. Even though the second term formally arises in a higher order of our expansion than the first, there is no reason in the world why λ cannot be of the same order of magnitude as e^4. Indeed, this is what we should expect if we think of the quartic meson self-interaction as being forced on us by renormalization, to cancel the divergence in Coulomb scattering, itself of order e^4.

We think this point is so important that we will restate it in slightly different language: At first glance, the central idea of this paper, that higher-order effects may qualitatively change the char-

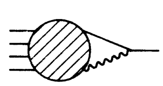

FIG. 3. Some diagrams which do not contribute to the effective potential in scalar quantum electrodynamics. The wiggly lines represent photons, the solid lines, spinless mesons.

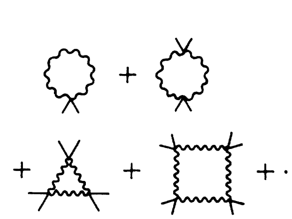

FIG. 4. The photon contribution to the one-loop approximation for the effective potential.

acter of a physical theory, seems to be in flat contradiction with the basic idea of perturbation theory, that higher-order effects are always small corrections to lower-order ones. This is not so. In the case under consideration, the first effects of the electromagnetic coupling, the radiative corrections (Fig. 3) are *pro forma* of higher order than the effects of the direct quartic coupling (Fig. 1). However, this says *nothing* about their actual relative magnitudes, which can be comparable, even for very small e and λ.

Therefore, for the time being, we will restrict ourselves to the case where λ is indeed of order e^4. In Sec. V, we shall use the renormalization group to show that all of our results extend without alteration to a more general case, arbitrary (but still small) e and λ.

Under this assumption, the term of order λ^2 in Eq. (4.5) is negligible compared to the other terms, and we can drop it. Indeed, consistency demands we drop it, since it is of the same order of magnitude as the two-loop electromagnetic corrections, which we have not computed. Also, since the renormalization mass M is arbitrary, to the order in which we are working [see remark (4) at the end of Sec. III], we might as well simplify our computational task by choosing it to be the actual location of the minimum, $\langle\varphi\rangle$. Thus we obtain

$$V=\frac{\lambda}{4\,!}\varphi_c{}^4+\frac{3e^4}{64\pi^2}\varphi_c{}^4\left(\ln\frac{\varphi_c{}^2}{\langle\varphi\rangle^2}-\frac{25}{6}\right)\ . \tag{4.6}$$

Since $\langle\varphi\rangle$ is defined to be the minimum of V, we deduce

$$\begin{aligned}0&=V'(\langle\varphi\rangle)\\&=\left(\frac{\lambda}{6}-\frac{11e^4}{16\pi^2}\right)\langle\varphi\rangle^3\ ,\end{aligned} \tag{4.7}$$

or,

$$\lambda=\frac{33}{8\pi^2}e^4\ . \tag{4.8}$$

Note that the redefinition of the coupling constant associated with shifting the fields has led to a determination of λ in terms of e. Phrased more dramatically, after we have done the shift, the final value of λ is independent of the initial value of λ. This smells suspiciously like some sort of bootstrap or eigenvalue condition, but this is not the case: We have not lost any free parameters; we started out with two (e and λ), and we have ended up with two (e and $\langle\varphi\rangle$). What is slightly surprising is that we have traded a dimensionless parameter, λ, for a dimensional one, $\langle\varphi\rangle$. We call this phenomenon dimensional transmutation. The reader who has followed our arguments should realize that dimensional transmutation is an inevitable feature of spontaneous symmetry breakdown in a massless field theory; it is nothing but a reflection of the simple fact that, for a fixed theory, a change in the arbitrary renormalization mass leads to a change in the numerical value of the dimensionless coupling constants.

Thus, we obtain our final expression for the effective potential

$$V=\frac{3e^4}{64\pi^2}\varphi_c{}^4\left(\ln\frac{\varphi_c{}^2}{\langle\varphi\rangle^2}-\frac{1}{2}\right)\ . \tag{4.9}$$

This is parametrized in terms of e and $\langle\varphi\rangle$ alone; all reference to λ has disappeared. If we had adapted a different definition of λ, the intermediate equations (4.5)–(4.8) would have changed, but Eq. (4.9) would have remained unaltered.

From this point on the analysis is the same as for the familiar Abelian Higgs model.[5] After shifting the field, the mass of the scalar meson is given by

$$\begin{aligned}m^2(S)&=V''(\langle\varphi\rangle)\\&=\frac{3e^4}{8\pi^2}\langle\varphi\rangle^2\ .\end{aligned} \tag{4.10}$$

The would-be Goldstone boson combines with the photon to make a massive vector meson; its mass is given by the conventional formula

$$m^2(V)=e^2\langle\varphi\rangle^2\ . \tag{4.11}$$

Combining these two equations, we find

$$\frac{m^2(S)}{m^2(V)}=\frac{3}{2\pi}\ \frac{e^2}{4\pi}\ , \tag{4.12}$$

the result announced in the Introduction. We emphasize that Eqs. (4.8)–(4.11) are exact only in our approximation. Their right-hand sides have corrections of higher order in e. Like all other radiative corrections, these are unambiguously calculable in principle, although, as the number of loops increases, they grow rapidly more difficult to compute in practice. Also, if we were to go to higher orders, we would have to define particle masses as the locations of poles in propagators rather than as the values of inverse propagators at zero momentum, as we have been doing. This is because the first of these definitions is gauge-invariant, while the second, in general, is not. Fortunately, to the order in which we are working, the two definitions coincide.

To what extent do we expect these yet higher-order corrections to affect the qualitative behavior we have found? Near $\langle\varphi\rangle$, $\ln(\varphi_c/\langle\varphi\rangle)$ is small, so we expect the effects of higher orders on the effective potential in this region to be small, despite the fact that graphs with more loops introduce higher powers of $\ln(\varphi_c/\langle\varphi\rangle)$ into the potential along with higher powers of e. For extremely large values of φ_c, where the logarithm becomes large, our entire expansion scheme breaks down. There may, perhaps, be new minima lurking in this region, but it is beyond the capabilities of any perturbative computational method to detect them.

Of course, logarithms grow large for small values of their arguments as well as large, so our expansion scheme is as unreliable near the origin as it is for large φ_c. Thus, although we should trust the minimum we have found at $\langle\varphi\rangle$, we should *not* trust the maximum we have found at zero; graphs with more loops might well turn this into a minimum again. However, no graph can change the value of V at zero, which stays firmly fixed at zero. On the other hand,

$$V(\langle\varphi\rangle) = -\frac{3e^4\langle\varphi\rangle^4}{128\pi^2}, \tag{4.13}$$

plus corrections which we expect to be small. Therefore, although graphs with more loops might turn the maximum at the origin back into a minimum, they cannot turn it into an absolute minimum, and the asymmetric vacuum we have found remains a local minimum definitely lower than that at the origin.

V. THE RENORMALIZATION GROUP

For simplicity, let us begin by restricting ourselves to the theory of a self-interacting meson field, discussed in Sec. III; later we will extend our analysis to massless quantum electrodynamics. As we have seen, the explicit expression for the effective action in this theory involves a mass, M. But, as we have emphasized, this mass is arbitrary; its only function is to define the renormalized coupling constant, λ, through Eq. (3.7), and the scale of the renormalized field, through Eq. (3.8). Thus, a small change in M can always be compensated for by an appropriate small change in λ and an appropriate small rescaling of the field. This statement can be expressed as an equation

$$\left[M\frac{\partial}{\partial M}+\beta\frac{\partial}{\partial\lambda}+\gamma\int d^4x\,\varphi_c(x)\frac{\delta}{\delta\varphi_c(x)}\right]\Gamma=0, \tag{5.1}$$

for an appropriate choice of the coefficients β and γ. (By dimensional analysis, β and γ can depend only on λ.)

If we apply Eq. (5.1) to the expansion of Γ in terms of 1PI Green's functions, Eq. (2.7), we find that

$$\left(M\frac{\partial}{\partial M}+\beta\frac{\partial}{\partial\lambda}+n\gamma\right)\Gamma^{(n)}(x_1\cdots x_n)=0. \tag{5.2}$$

These are the familiar differential equations of the renormalization group. This should be no surprise, for the argument given in the preceding paragraph is just the standard argument used to derive the renormalization group.[9]

For our purposes, though, it is better to exploit Eq. (5.1) by applying it to the expansion (2.8). In this way we obtain

$$\left(M\frac{\partial}{\partial M}+\beta\frac{\partial}{\partial\lambda}+\gamma\varphi_c\frac{\partial}{\partial\varphi_c}\right)V=0 \tag{5.3}$$

and

$$\left(M\frac{\partial}{\partial M}+\beta\frac{\partial}{\partial\lambda}+\gamma\varphi_c\frac{\partial}{\partial\varphi_c}+2\gamma\right)Z=0. \tag{5.4}$$

It will turn out to be convenient, instead of working with V, to work with the dimensionless function

$$V^{(4)}=\frac{\partial^4 V}{\partial\varphi_c{}^4}. \tag{5.5}$$

Since dimensionless quantities, such as $V^{(4)}$ and Z, can only depend on φ_c and M through the ratio φ_c/M, it is convenient to define

$$t=\ln(\varphi_c/M). \tag{5.6}$$

It is also useful to define

$$\bar\beta=\beta/(1-\gamma), \tag{5.7}$$

and

$$\bar\gamma=\gamma/(1-\gamma). \tag{5.8}$$

In terms of these quantities, Eqs. (5.3) and (5.4) become

$$\left(-\frac{\partial}{\partial t}+\bar\beta\frac{\partial}{\partial\lambda}+4\bar\gamma\right)V^{(4)}(t,\lambda)=0 \tag{5.9}$$

and

$$\left(-\frac{\partial}{\partial t}+\bar\beta\frac{\partial}{\partial\lambda}+2\bar\gamma\right)Z(t,\lambda)=0. \tag{5.10}$$

This is the form of the renormalization-group equations that we shall use.

In terms of the quantities we have defined, the renormalization conditions, (3.7) and (3.8), become

$$V^{(4)}(0,\lambda)=\lambda \tag{5.11}$$

and

$$Z(0,\lambda)=1. \tag{5.12}$$

If we combine these with Eqs. (5.9) and (5.10), we find that

$$\bar\gamma=\frac{1}{2}\frac{\partial}{\partial t}Z(0,\lambda), \tag{5.13}$$

and

$$\bar{\beta}=\frac{\partial}{\partial t}V^{(4)}(0,\lambda)-4\bar{\gamma}\lambda\ . \tag{5.14}$$

Thus, if we know the derivatives that occur on the right-hand sides of these equations, we know $\bar{\beta}$ and $\bar{\gamma}$. Of course, even if we are very industrious, we cannot know these derivatives exactly; at best, we can only know the first few terms in their loop expansions. Thus, we can only know the first few terms in the power-series expansions of $\bar{\beta}$ and $\bar{\gamma}$. Nevertheless, as we shall see, even this is useful information.

For the moment, though, let us imagine we know $\bar{\beta}$ and $\bar{\gamma}$ exactly. Then it is easy to construct the general solution of the equation

$$\left(-\frac{\partial}{\partial t}+\bar{\beta}\frac{\partial}{\partial\lambda}+n\bar{\gamma}\right)F(t,\lambda)=0\ , \tag{5.15}$$

of which Eqs. (5.9) and (5.10) are special cases. Since this is of the same structure as the much-studied Eq. (5.2), we will merely describe the construction, and refer the reader to the literature[10] for its derivation. We begin by finding the function $\lambda'(t,\lambda)$ defined as the solution of the ordinary differential equation

$$\frac{d\lambda'}{dt}=\bar{\beta}(\lambda')\ , \tag{5.16}$$

with the boundary condition

$$\lambda'(0,\lambda)=\lambda\ . \tag{5.17}$$

Then, the general solution of Eq. (5.15) is of the form

$$F(t,\lambda)=f(\lambda'(t,\lambda))\exp\left[n\int_0^t dt\ \bar{\gamma}(\lambda'(t,\lambda))\right]\ , \tag{5.18}$$

where f is an arbitrary function.

For the special cases of Z and $V^{(4)}$, the renormalization equations, (5.11) and (5.12), fix the arbitrary function f. Thus we obtain

$$Z(t,\lambda)=\exp\left[2\int_0^t dt\ \bar{\gamma}(\lambda'(t,\lambda))\right]\ , \tag{5.19}$$

and

$$V^{(4)}(t,\lambda)=\lambda'(t,\lambda)[Z(t,\lambda)]^2\ . \tag{5.20}$$

This is a remarkable result, and shows the power of the renormalization group; Z and $V^{(4)}$ are completely determined, for all t, in terms of $\bar{\beta}$ and $\bar{\gamma}$, that is to say, in terms of their first derivatives at $t=0$.

We can go further, and use Eqs. (5.19) and (5.20) to compute how the renormalized coupling constant changes when we change the renormalization mass. Let us suppose we change our renormalization point from $\varphi_c=M$ to

$$\varphi_c=M'\ . \tag{5.21}$$

If φ' is the new field, then

$$(\partial_\mu\varphi_c')^2=(\partial_\mu\varphi_c)^2 Z(\ln(M'/M),\lambda)\ , \tag{5.22}$$

whence

$$\varphi_c'=\varphi_c Z(\ln(M'/M),\lambda)^{1/2}\ . \tag{5.23}$$

By definition, the new coupling constant, λ', is given by

$$\begin{aligned}\lambda'&=\left.\frac{\partial^4 V}{\partial\varphi_c'^4}\right|_{\varphi_c=M'}\\&=Z^{-2}V^{(4)}(\ln(M'/M),\lambda)\\&=\lambda'(\ln(M'/M),\lambda)\ .\end{aligned} \tag{5.24}$$

This elucidates the meaning of the function $\lambda'(t,\lambda)$.

It is easy to extend all this analysis to a general massless renormalizable field theory: The renormalization group equations will have one β-like term for every coupling constant and one γ-like term for every field. Equation (5.16) will become a system of coupled ordinary differential equations, one for each coupling constant. Otherwise, all will be as before; those functions in the effective action that fix the renormalization conditions will be determined in terms of their derivatives at the renormalization point, and we will be able to trace out the changes in the renormalized coupling constants as the renormalization point changes.

So much for the dream world in which we know β and γ exactly. What about the real world, in which we know them only to lowest order? Let us suppose that we construct an approximation for $\lambda'(\lambda,t)$ by integrating, from some small λ, an approximate version of Eq. (5.16), obtained by replacing $\bar{\beta}$ by its one-loop approximation. We would expect this approximation to be reliable for that range of t for which $\lambda'(t,\lambda)$ remains small, for, in this domain, the terms we have neglected by truncating the power series for $\bar{\beta}$ are indeed small compared to the terms we have retained. However, if, for some range of t, $\lambda'(t,\lambda)$ becomes large, the approximation becomes evident nonsense, for then the terms we have neglected are large compared to the terms we have retained. If we are lucky, the approximation for λ' will stay small for a large range of t, and we will obtain an improved approximation – "improved" in the sense that it is valid over a larger range of t. (It is important to remember that the domain of reliability of the one-loop approximation is determined not only by the condition that the coupling constant be small, but also by the condition that the logarithmic factor t not be too large. It is the latter restraint that the renormalization group may allow us to escape, not the former.)

Let us apply these ideas to the one-loop approximation in self-interacting meson theory, discussed in Sec. III. Differentiating Eq. (3.10), we find that

$$V^{(4)}=\lambda+\frac{3\lambda^2 t}{16\pi^2}\,, \tag{5.25}$$

in the one-loop approximation. In Sec. III, we did not compute the one-loop corrections to Z; however, the computation is straightforward, and the result is that, in the one-loop approximation, they vanish:

$$Z=1\,. \tag{5.26}$$

Thus, in this approximation,

$$\bar{\gamma}=0 \tag{5.27}$$

and

$$\bar{\beta}=\frac{3\lambda^2}{16\pi^2}\,. \tag{5.28}$$

The approximate differential equation is

$$\frac{d\lambda'}{dt}=\frac{3\lambda'^2}{16\pi^2}\,. \tag{5.29}$$

Its solution is

$$\lambda'=\frac{\lambda}{1-3\lambda t/16\pi^2}\,. \tag{5.30}$$

From this we obtain the improved approximation for the effective potential,

$$V^{(4)}=\frac{\lambda}{1-3\lambda t/16\pi^2}\,. \tag{5.31}$$

Note that this agrees with the one-loop approximation in its expected domain of validity ($|\lambda|\ll 1$, $|\lambda t|\ll 1$). However, Eq. (5.31) is valid in a much larger range of t, including all negative t, for in this range λ' remains small. In Sec. III, we found that the one-loop corrections turned the minimum at the origin of classical-field space into a maximum, but we mistrusted this, because the region near the origin (large negative t) was outside the domain of validity of the approximation. We now see that our mistrust was justified; Eq. (5.31) *is* a good approximation near the origin, and it predicts a maximum, not a minimum. In Sec. III we also found a phony minimum at large t; in place of this, Eq. (5.31) has a pole at large t, but it is equally phony – this is a region of large λ', where our new approximation is as untrustworthy as our old one.

We now turn to scalar electrodynamics. We begin by expanding the effective action and retaining only those terms which define the renormalization constants:

$$\Gamma=\int d^4x\{-V(\varphi_c)-\tfrac{1}{4}H(\varphi_c)(F_{\mu\nu c})^2+\tfrac{1}{2}Z(\varphi_c)[(\partial_\mu\varphi_{1c}-eA_{\mu c}\varphi_{2c})^2+(\partial_\mu\varphi_{2c}+eA_{\mu c}\varphi_{1c})^2]+\cdots\}\,. \tag{5.32}$$

Note that the last group of terms has a common coefficient function, Z; this is a consequence of gauge invariance. We impose the same renormalization conditions on Z and V as before; in addition, we fix the scale of the renormalized electromagnetic field by

$$H(M)=1\,. \tag{5.33}$$

We have computed all these functions in the one-loop approximation. The results are

$$V^{(4)}=\lambda+\left(\frac{5\lambda^2}{24\pi^2}+\frac{9e^4}{4\pi^2}\right)t, \tag{5.34}$$

$$Z=1+\frac{3e^2}{8\pi^2}t\,, \tag{5.35}$$

and

$$H=1-\frac{e^2}{24\pi^2}t\,, \tag{5.36}$$

where t, as before, is $\ln(\varphi_c/M)$.

The renormalization-group equation for this theory is

$$\left[M\frac{\partial}{\partial M}+\beta\frac{\partial}{\partial\lambda}+\beta_e\frac{\partial}{\partial e}+\gamma_e\int A_{\mu c}\frac{\delta}{\delta A_{\mu c}}+\gamma\left(\int\varphi_{1c}\frac{\delta}{\delta\varphi_{1c}}+\int\varphi_{2c}\frac{\delta}{\delta\varphi_{2c}}\right)\right]\Gamma=0\,. \tag{5.37}$$

Applying this to Eq. (5.32), we deduce that

$$\beta_e=-e\gamma_e\,. \tag{5.38}$$

This is the reflection in our formalism of the old result that $Z_1=Z_2$. We also find that

$$\left(-\frac{\partial}{\partial t}+\bar{\beta}\frac{\partial}{\partial\lambda}+\bar{\beta}_e\frac{\partial}{\partial e}+4\bar{\gamma}\right)V^{(4)}(t,\lambda,e)=0\,, \tag{5.39}$$

$$\left(-\frac{\partial}{\partial t}+\bar{\beta}\frac{\partial}{\partial\lambda}+\bar{\beta}_e\frac{\partial}{\partial e}+2\bar{\gamma}\right)Z(t,\lambda,e)=0\,, \tag{5.40}$$

and

$$\left(-\frac{\partial}{\partial t}+\bar{\beta}\frac{\partial}{\partial\lambda}+\bar{\beta}_e\frac{\partial}{\partial e}+2\bar{\gamma}\right)H(t,\lambda,e)=0\,. \tag{5.41}$$

Here the barred coefficients are defined as before, by dividing by $(1-\gamma)$.

Evaluating these equations at the renormalization point, we obtain the one-loop approximation to the coefficient functions:

$$\bar{\gamma}=3e^2/16\pi^2 \ , \tag{5.42}$$

$$\bar{\beta}=(\tfrac{5}{6}\lambda^2-3e^2\lambda+9e^4)/4\pi^2 \ , \tag{5.43}$$

and

$$\bar{\beta}_e=e^3/48\pi^2 \ . \tag{5.44}$$

Thus, the approximate differential equations we must solve are

$$\frac{de'}{dt}=e'^3/48\pi^2 \ , \tag{5.45}$$

and

$$\frac{d\lambda'}{dt}=(\tfrac{5}{6}\lambda'^2-3e'^2\lambda'+9e'^4)/4\pi^2 \ . \tag{5.46}$$

The first of these can be solved trivially, by quadrature:

$$e'^2=\frac{e^2}{1-e^2t/24\pi^2} \ . \tag{5.47}$$

The second looks more fearsome; however, if we define

$$R=\lambda'/e'^2 \ , \tag{5.48}$$

Eq. (5.46) becomes

$$e'^2\frac{dR}{d(e'^2)}=5R^2-19R+54 \ . \tag{5.49}$$

This can also be solved by quadrature:

$$\lambda'=\tfrac{1}{10}e'^2[\sqrt{719}\tan(\tfrac{1}{2}\sqrt{719}\ln e'^2+\theta)+19] \ , \tag{5.50}$$

where θ is an integration constant, to be chosen such that $\lambda'=\lambda$ when $e'=e$.

We can hardly praise these solutions for their beauty. What about their utility? As t becomes large and positive, e' becomes large, so our approximation certainly breaks down in this region, just as in the preceding case. As t becomes large and negative, e' stays small, but λ' becomes large, since the argument of the tangent inevitably passes through a multiple of π. Thus our approximation also breaks down in this region; unlike the preceding case, we cannot use the renormalization group to obtain an approximation to the effective potential that is valid near the origin of classical-field space as well as near the renormalization point.

Nevertheless, all is not lost; we can still obtain useful information. For we can make the argument of the tangent change by 2π by varying t so as to move e'^2 through a very small range. (By the crudest estimates, the range from $\frac{1}{2}e^2$ to $2e^2$ is more than sufficient.) In the course of this variation, λ' traverses the entire real axis. Of course, we cannot trust the further reaches of this excursion, but we can trust it for the region of small λ'.

Thus, even for very small e, we can move λ from any small value to any other small value by a change in the renormalization mass that does not change the order of magnitude of e. This means that the restriction we imposed in Sec. IV – that λ be of the order of magnitude of e^4 – is in fact no restriction at all. For, if λ is not of the order of magnitude of e^4, we can always make it such, by appropriately changing the renormalization mass. Thus, the effective potential for massless scalar electrodynamics develops a minimum away from the origin, and spontaneous symmetry breakdown occurs, for arbitrary small e and λ.

VI. NON-ABELIAN GAUGE THEORIES

In this section we compute in closed form the one-loop corrections to the effective potential in a general massless renormalizable gauge-field theory. The expressions we obtain involve traces of functions of certain matrices constructed from the coupling constants of the theory. For any given theory, it is simple to compute these quantities, and it is then a straightforward calculus exercise to find the minima of V. Unfortunately, we do not have enough skill in the manipulation of arbitrary representations of arbitrary gauge groups to be able to give even a qualitative discussion of the structure of spontaneous symmetry breaking in the general case; therefore we restrict our detailed discussion to three specific models for which we have done explicit computations. One is a Yang-Mills triplet coupled to a scalar isovector; another is the same triplet coupled to a scalar isotensor; the third is a massless version of the Weinberg-Salam theory of leptons. These display a range of interesting phenomena, but in their gross features they are all the same as massless scalar electrodynamics: Radiative corrections induce spontaneous symmetry breaking, and all mass ratios are computable in terms of dimensionless coupling constants.

We should emphasize that all of our analysis is on the level of that of Sec. IV. In particular, this means that we will always assume that the quartic scalar self-couplings are of the order of magnitude of e^4. In Sec. V we were able to remove this restriction from our study of massless scalar electrodynamics with the aid of the renormalization group, but we have not yet extended these methods to the non-Abelian case.

A. Computation of the Effective Potential

The class of Lagrangians we will study involves a set of real spinless boson fields, which we denote by φ^a, a set of Dirac bispinor fields, which we denote by Ψ^a, and a set of real vector fields, which we denote by A^a_μ. The index a runs over the appropriate range in each case. Sometimes we will find it convenient to assemble all the spinless fields into a vector, which we denote by $\vec{\varphi}$. All these fields are massless, and the interactions between them are of renormalizable type: quartic self-interactions of the spinless bosons, Yukawa-type boson-fermion couplings (not necessarily parity-conserving), and minimal gauge-invariant couplings of the vector fields.

If we quantize the theory in Landau gauge, the only graphs we need consider are polygon graphs with either spinless mesons (Fig. 2), gauge mesons (Fig. 4), or fermions (Fig. 5) running around the loops. Note that in the fermion sum, only graphs with an even number of external lines occur; this is because the trace of an odd number of γ matrices vanishes. These are not the only internal lines in a non-Abelian gauge theory; there are also the famous ghost fields.[11] However, in Landau gauge, these have no direct coupling to the spinless fields, and thus need not be considered for our computation. (Of course, we would have to take account of ghost fields if we were to compute terms in the effective action that depend on classical gauge fields, or if we were to compute two-loop corrections to the effective potential, or even if we were to work in a less well-chosen gauge.)

Thus, the one-loop approximation to the effective potential can be written as a sum of terms:

$$V(\vec{\varphi}_c) = V_0 + V_s + V_f + V_g + V_c \ , \tag{6.1}$$

where V_0 is the zero-loop approximation, V_s, V_f, and V_g are the contributions from spinless-meson, fermion, and gauge-field loops, and V_c is the contribution from coupling-constant-renormalization counterterms. V_c is a quartic polynomial determined in terms of the other four terms once we have stated our renormalization conditions.

To compute V_s, we need to know the vertex that occurs at the corner of a polygon graph, connecting an internal meson of type a with one of type b. This vertex, shown in Fig. 6, is given by

$$\underline{W}_{ab}(\vec{\varphi}_c) = \frac{\partial^2 V_0}{\partial\varphi_a \partial\varphi_b} \ . \tag{6.2}$$

$\underline{W}$ is a real symmetric matrix and a quadratic function of $\vec{\varphi}_c$. To compute the n-sided polygon, we must sum over all possible internal mesons. This corresponds to multiplying the W matrices around the loop and then taking their trace. Thus, copying directly from Sec. III, we find that

$$V_s = \frac{1}{64\pi^2} \mathrm{Tr}[\underline{W}^2(\vec{\varphi}_c) \ln \underline{W}(\vec{\varphi}_c)] \ , \tag{6.3}$$

plus cutoff-dependent quartic terms, which we absorb in V_c.

The contribution of the gauge-field loops may be computed in a similar way. We define a matrix $\underline{M}^2(\vec{\varphi})$ in terms of the nonderivative coupling of gauge fields to spinless mesons:

$$\mathcal{L} = \cdots + \tfrac{1}{2} \sum_{ab} \underline{M}^2(\vec{\varphi}) A_{\mu a} A^\mu_b + \cdots \ . \tag{6.4}$$

Like $\underline{W}$, $\underline{M}^2$ is a real symmetric matrix and a quadratic function of $\vec{\varphi}$. We call this matrix $\underline{M}^2$ because, to first nonvanishing order $\underline{M}^2$, $(\langle\vec{\varphi}\rangle)$ is the square of the vector-meson mass matrix. Because the vector mesons are minimally coupled, we can write

$$\underline{M}^2{}_{ab} = g_a g_b (\underline{T}_a \vec{\varphi}, \underline{T}_b \vec{\varphi}) \ , \tag{6.5}$$

where $\underline{T}_a$ is the representation of the ath infinitesimal transformation of the gauge group, and g_a is the coupling constant of the associated gauge field. (If the gauge group is simple, all the g's are equal; otherwise, this is not necessarily the case.) Following the same reasoning as before,

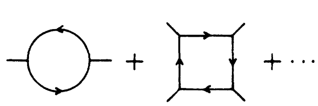

FIG. 5. The fermion contribution to the effective potential. The directed lines represent fermions.

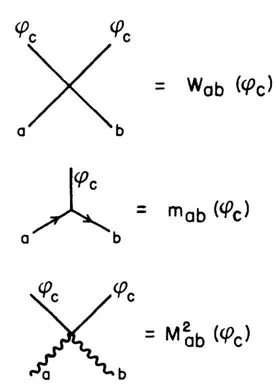

FIG. 6. The vertices which are needed to compute the one-loop approximation to $V(\varphi_c)$ in a general massless gauge theory. These are to be inserted in the vertices of the polygon graphs of Figs. 2, 4, and 5.

we find that

$$V_g = \frac{3}{64\pi^2} \mathrm{Tr}[\underline{M}^4(\vec{\varphi}_c) \ln \underline{M}^2(\vec{\varphi}_c)] \ , \tag{6.6}$$

plus quartic terms, which we absorb in V_c. The extra factor of three in this equation comes from the trace of the numerator of the Landau-gauge propagator.

The contribution of the fermion loops may be computed in a similar way. We define a matrix $\underline{m}(\vec{\varphi})$ in terms of the Yukawa coupling:

$$\mathcal{L} = \cdots - \sum_{ab} \bar{\Psi}_a \underline{m}_{ab}(\vec{\varphi}) \Psi_b + \cdots \ . \tag{6.7}$$

The matrix $\underline{m}$ is a matrix in Dirac space as well as in internal space:

$$\underline{m}_{ab} = \underline{A}_{ab} + i \underline{B}_{ab} \gamma_5 \ . \tag{6.8}$$

(We use a Hermitian γ_5.) $\underline{A}$ and $\underline{B}$ are Hermitian matrices and linear functions of $\vec{\varphi}$. To first nonvanishing order, $\underline{m}(\langle\vec{\varphi}\rangle)$ is the fermion mass matrix, whence its name. We can exploit the fact that only graphs with an even number of internal fermions contribute to the sum by grouping terms in the matrix product pairwise:

$$\cdots \underline{m} \frac{1}{k\!\!\!/} \underline{m} \frac{1}{k\!\!\!/} \cdots = \cdots \frac{1}{k^2} \underline{m}\, \underline{m}^\dagger \cdots \ . \tag{6.9}$$

The sum is then seen to be the same as in the other cases, except, of course, for the standard minus sign for fermion loops,[12] and we obtain

$$V_f = -\frac{1}{64\pi^2} \mathrm{Tr}[(\underline{m}\,\underline{m}^\dagger(\vec{\varphi}_c))^2 \ln(\underline{m}\,\underline{m}^\dagger(\vec{\varphi}_c))] \ , \tag{6.10}$$

plus quartic terms, which we absorb in V_c. Note that in this equation the trace runs over Dirac indices as well as internal indices.

The fermion contribution to the effective potential has opposite sign to the other terms; this means that the effect of fermion closed loops can make shallower, or even eliminate altogether, minima caused by the other terms. However, for the only model with fermions we shall consider (the Weinberg-Salam theory of leptons), this effect is completely negligible, because, as we see from Eqs. (6.6) and (6.10), V_f is smaller than V_g by a factor on the order of the fourth power of the fermion-to-vector-meson mass ratio. For this theory, this ratio is so small that the effect of fermion loops is tiny even when compared with that of two-loop electromagnetic corrections. We should bear in mind, though, that if we build a model in which the Yukawa couplings are such that superheavy fermions appear, with masses comparable to the vector-meson masses, this effect can be important.

B. First Model: Yang-Mills Fields and a Scalar Isovector

We now turn to our first model, the theory of a triplet of SU(2) gauge fields minimally coupled to a set of scalar mesons transforming according to the vector representation of the group. We assemble the scalar fields into a vector in the standard way. By SU(2) invariance, the effective potential can depend only on the length of this vector; therefore it suffices to compute it in the case when only the third component is nonzero:

$$(\varphi_{1c}, \varphi_{2c}, \varphi_{3c}) = (0, 0, \varphi_c) \ . \tag{6.11}$$

We begin by computing V_g, since, from our experience with the Abelian theory, we expect this to be the dominant term in the effective potential. From Eq. (6.5) we find that

$$\underline{M}^2{}_{11} = \underline{M}^2{}_{22} = e^2 \ , \tag{6.12}$$

where e is the gauge-field coupling constant; all other entries vanish. Thus we find, from Eq. (6.6), that

$$V_g = \frac{3e^4}{32\pi^2} \varphi_c{}^4 \ln \varphi_c{}^2 \ . \tag{6.13}$$

This is, except for the coefficient, of exactly the same form as the corresponding term in massless scalar electrodynamics, Eq. (4.5). Thus, the discussion of its minimum, and of the effects of shifting the renormalization point to the minimum, is the same as that given in Sec. III.

Hence, spontaneous symmetry breakdown occurs, and the field develops a vacuum expectation value. The vacuum expectation value is necessarily invariant under a U(1) subgroup of the original SU(2) group, which we identify with electric charge. The gauge field associated with this subgroup (the photon) remains massless, while the two other (charged) gauge fields eat the corresponding scalar mesons and acquire a mass. The ratio of the masses of the charged vector mesons to that of the remaining (neutral) scalar meson is given by

$$\frac{m^2(S)}{m^2(V)} = \frac{3}{\pi} \frac{e^2}{4\pi} \ , \tag{6.14}$$

and the quartic scalar coupling constant is given by

$$\lambda = \frac{33 e^4}{4\pi^2} \ . \tag{6.15}$$

Because the potential is twice as great as that of the Abelian theory, these formulas differ by a factor of two from the corresponding ones we found in Sec. IV. Since in this model a massless photon remains after symmetry breakdown, it is not foolish to identify e with the actual electromagnetic coupling constant; if we do this, Eq. (6.14) predicts that the charged vector mesons are about twelve times more massive than the neutral scalar meson.

C. Second Model: Yang-Mills Fields and a Scalar Isotensor

Our second model is the same as the first, except that the scalar fields transform according to the five-dimensional (tensor) representation of SU(2).[13] We represent these in the standard way as a traceless symmetric matrix. In computing the effective potential, we can, with no loss of generality, choose this matrix to be diagonal:

$$\underline{\varphi}_c = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \tag{6.16}$$

where

$$a + b + c = 0 \ . \tag{6.17}$$

We would be especially happy if the minimum of the effective potential corresponded to a situation for which

$$a = b = -\tfrac{1}{2}c \ . \tag{6.18}$$

For, in this case, there would remain an unbroken U(1) subgroup of the original SU(2) group, which we could identify with electric charge, and associated with this subgroup, a massless vector meson, which we could identify with the photon. The other particles remaining in the theory would be a pair of charged vector mesons, a neutral scalar, and a pair of doubly charged scalars.

We emphasize that there is nothing in the symmetry properties of this theory that guarantees that the minimum of V will obey Eq. (6.18). Thus, in contrast to the preceding model, if a massless photon emerges, it will be as a consequence of detailed dynamics, not just of trivial group theory.

We now turn to the actual computation of the effective potential. We begin by investigating the restrictions placed by SU(2) symmetry upon the zeroth-order potential V_0. In contrast to the previous case, a cubic interaction, $\mathrm{Tr}\underline{\varphi}^3$, is allowed by the group. To simplify our problem, we exclude this by adding to our continuous symmetry group the discrete transformation $\underline{\varphi} \to -\underline{\varphi}$. There are apparently two possible quartic couplings, $\mathrm{Tr}\underline{\varphi}^4$ and $(\mathrm{Tr}\underline{\varphi}^2)^2$; however, these are related by the tracelessness of $\underline{\varphi}$:

$$\mathrm{Tr}\underline{\varphi}^4 = \tfrac{1}{2}(\mathrm{Tr}\underline{\varphi}^2)^2 \ . \tag{6.19}$$

Thus there is only one term in V_0.

[Digression: This has an amusing consequence, for even if we had destabilized the vacuum by hand, by introducing a negative mass term in the Lagrangian, V_0 would still possess a larger symmetry than SU(2), to wit, SO(5).[14] Thus, even in this case, if we analyzed the theory in the semiclassical (no loop) approximation, we would be in trouble for two reasons: (1) We would only determine $a^2 + b^2 + c^2$ at the minimum, and would have no idea how to fix the other SU(2) invariant, abc; (2) even if we picked a vacuum expectation value arbitrarily from this over-rich set, we would find that the doubly charged scalars were spurious Goldstone bosons, and had zero mass. To pick the right vacuum, and to obtain a nonzero mass for the doubly charged scalars, it would be necessary to compute the effects of gauge-field loops. Moral: The techniques developed in this paper may be of use even to someone who thinks that all this talk of massless scalars is nonsense.]

To compute V_g, we begin by calculating the vector-meson mass-squared matrix:

$$M^2 = 2e^2 \begin{pmatrix} (b-c)^2 & 0 & 0 \\ 0 & (c-a)^2 & 0 \\ 0 & 0 & (a-b)^2 \end{pmatrix}. \tag{6.20}$$

Hence,

$$V_g = \frac{3e^4}{16\pi^2}(a-b)^4 \ln[(a-b)^2/\mu^2] + \text{cyclic permutations}, \tag{6.21}$$

where μ^2 is an arbitrary renormalization mass. Just as before, a change in μ^2 can always be compensated for by a change in the quartic coupling constant, λ. Just as in Sec. IV, we now assume that λ is of the order of magnitude of e^4; then V_s is small compared with V_g, and we can neglect it. Further, we can always adjust μ^2 to give the quartic coupling any desired value; we choose to do this such that

$$\begin{aligned} V &= V_0 + V_g \\ &= \frac{3e^4}{16\pi^2}(a-b)^4\left[\ln\frac{(a-b)^2}{\mu^2} - \frac{1}{2}\right] \\ &\quad + \text{cyclic permutations}. \end{aligned} \tag{6.22}$$

From this expression it is evident that

$$\begin{aligned} a = b &= \tfrac{1}{3}\mu\,, \\ c &= -\tfrac{2}{3}\mu\,, \end{aligned} \tag{6.23}$$

is a minimum of V, for it is a minimum of two of the three terms in Eq. (6.22) and a maximum (but with vanishing second derivative) of the third. It is possible to show that this is the only minimum of V (other than those obtained from it by trivial permutations), but we will not give the proof here. Note that this is of the form (6.18); thus, electric charge remains a manifest symmetry and there is a massless photon.

The mass of the charged vector follows directly from (6.20):

$$m^2(V)=2e^2\mu^2 . \tag{6.24}$$

Fortunately, to compute the scalar masses, we need only consider diagonal perturbations in Eq. (6.16):

$$\begin{aligned} a&=\frac{\mu}{3}+\frac{S}{\sqrt{6}}+\frac{D}{\sqrt{2}} , \\ b&=\frac{\mu}{3}+\frac{S}{\sqrt{6}}-\frac{D}{\sqrt{2}} , \\ c&=-\frac{2\mu}{3}-\frac{2S}{\sqrt{6}} , \end{aligned} \tag{6.25}$$

where S is the neutral scalar and D is a Hermitian combination of the two charged scalars. [The square roots are to give these fields the right normalization in the kinetic energy, $\frac{1}{2}\,\mathrm{Tr}(\partial_\mu \underline{\varphi}\partial^\mu \underline{\varphi})$.] Expanding (6.22), we find

$$m^2(S)=9e^4\mu^2/2\pi^2 , \tag{6.26}$$

and

$$m^2(D)=3e^4\mu^2/2\pi^2 . \tag{6.27}$$

Whence,

$$\frac{m^2(S)}{m^2(D)}=3 , \tag{6.28}$$

and

$$\frac{m^2(D)}{m^2(V)}=\frac{3}{\pi}\frac{e^2}{4\pi} . \tag{6.29}$$

[We should note that had we put a negative mass term in the theory, as discussed in the preceding digression, Eq. (6.29) would still have survived, because the no-loop approximation would still have made no contribution to the mass of the doubly charged scalars.]

D. Weinberg-Salam Theory of Leptons

The model of leptons proposed by Weinberg and Salam[15] has been so much discussed in the recent literature that, rather than explaining it in detail, we shall just remind the reader of some of its salient features. The gauge group is SU(2)×U(1); the coupling constant of the triplet gauge fields is called g, that of the singlet g'. The spinless mesons form a complex doublet; the Lagrangian contains a negative mass term for this doublet, which forces spontaneous symmetry breaking. Because of the symmetry, we can, with no loss of generality, arrange that only one component of the doublet has nonzero vacuum expectation value, and we can choose this value, which we denote by $\langle\varphi\rangle$, to be real. After symmetry breaking, the vector mesons are a massless photon, with coupling

$$e^2=\frac{g^2g'^2}{g^2+g'^2} , \tag{6.30}$$

two massive charged mesons, $W^\pm$, identified with the intermediate bosons of the weak interactions, with masses

$$m^2(W^\pm)=\tfrac{1}{4}g^2\langle\varphi\rangle^2 , \tag{6.31}$$

and a massive neutral meson, Z, with mass

$$m^2(Z)=\tfrac{1}{4}(g^2+g'^2)\langle\varphi\rangle^2 . \tag{6.32}$$

There is also a neutral scalar meson, S, the only surviving member of the original complex doublet; its mass is not determined in terms of the other masses in the theory. Also, of course, there are the leptons, but for our immediate purposes these are of no interest, since they are just the usual electron, muon, and neutrinos, and by the remark at the end of Sec. VI A, they make a negligible contribution to the effective potential.

We want to investigate what new information we obtain if we assume that, before symmetry breakdown, the scalar masses are zero, and that symmetry breakdown is driven by radiative corrections. Playing the game just as before, we see that the important thing is to compute V_g. Fortunately, the matrix $\underline{M}^2$ is given to us, in diagonal form, by Eqs. (6.31) and (6.32); all we have to do is to substitute φ_c for $\langle\varphi\rangle$. Thus we obtain

$$V_g=\frac{3}{1024\pi^2}[2g^4+(g^2+g'^2)^2]\varphi_c^4\ln\varphi_c^2 , \tag{6.33}$$

plus quartic terms, which are of no importance. Comparing this with our previous work, we immediately find

$$\begin{aligned} m^2(S)&=\frac{3}{128\pi^2}[2g^4+(g^2+g'^2)^2]\langle\varphi\rangle^2 \\ &=\frac{3}{32\pi^2}[2g^2m^2(W^\pm)+(g^2+g'^2)m^2(Z)] . \end{aligned} \tag{6.34}$$

The important observation is that this puny relation is *all* the new information we obtain. This is what we would expect from simple parameter counting. We have added one new assumption – vanishing scalar mass before symmetry breakdown – and have obtained one new relation – a prediction of the scalar mass after symmetry breakdown. This is obviously what will happen if we apply our ideas to any of the current horde of gauge theories of the weak and electromagnetic interactions. Since these theories typically either contradict experiment whatever the scalar masses, or have so many free parameters that they can be made to fit experiment whatever the scalar masses, we do not believe that the ideas developed in this paper will make a significant contribution to current weak-interaction phenomenology. Nevertheless, some day the grail may be found; theorists may discover a simple and compelling model with only a small number of adjustable parameters that fits experimental reality. If and when this happens, it will be interesting to see what constraints are put on the parameters of this model by the condition that the theory be fully massless.

VII. CONCLUSIONS AND A SPECULATION

(1) We hope that the reader who has followed our arguments will now agree with our basic contentions: that radiative corrections can be the dominant driving force of spontaneous symmetry breaking; that, at least for weak coupling constants, this possibility can be investigated in a systematic way; and that, in particular, massless scalar electrodynamics undergoes radiatively induced spontaneous symmetry breakdown.

We are very much aware that we are exploring unconventional ideas and that there may be some basic flaw in our whole approach which we have been too stupid to see. Barring this possibility, though, we believe we have done an honest computation by the standards of perturbation theory, being careful of our approximations and not discarding terms obviously large compared to those retained. We therefore feel that our computations have the same *a priori* plausibility as any other perturbative computation in a renormalizable field theory with weak coupling constants, such as, for example, the computation of the anomalous magnetic moment of the electron.

(2) Since our work does lead to the determination of some coupling constants in terms of others [for example, Eq. (4.8)], it is natural to ask what connection it has with the generalized eigenvalue condition,[16] which also seeks to determine coupling constants by demanding that, in the real world, the β coefficients in the renormalization-group equations vanish. The answer is that the two ideas cannot both be true. For, if the β's vanish, so do the $\bar{\beta}$'s, by Eq. (5.7). If the $\bar{\beta}$'s vanish, the effective potential is a simple power. If the effective potential is a simple power, then its only possible minimum is at the origin.

(3) As we explained at the end of Sec. VI, we do not believe that our ideas will be of any immediate use in the currently popular gauge theories of weak and electromagnetic interactions. These theories typically contain so many arbitrary parameters that the additional constraints imposed by demanding that the scalar mesons be massless before symmetry breakdown offer only a slight improvement. Nevertheless, if a model is found that satisfies experiment and has only a small number of free parameters, these constraints may play an important role. Meanwhile, as shown by the model of VI B, some of our computational techniques may be of use even in theories in which symmetry breakdown *is* driven by a negative mass term, to compute the masses of spurious Goldstone bosons.

(4) The speculation: A bold way of interpreting our results would be to say that nature abhors massless particles with long-range interactions between them, and escapes these abhorrent systems by the Goldstone-Higgs mechanism, which gives the particles a mass, or makes the interactions short-range, or both. If we accept this statement, then we should expect symmetry breakdown even in theories *without* fundamental spinless bosons, such as massless spinor electrodynamics, or the theory of non-Abelian gauge fields coupled to massless fermions.

Unfortunately, we know of no computational scheme, analogous to the one we have used, to study whether this in fact happens, even for weak coupling constants. The trouble is that the object which we would expect to develop a vacuum expectation value is a composite field, like $\bar{\Psi}\Psi$. In principle, there is no obstacle to extending the formalism of Sec. II to such objects: the formal machinery of the functional Legendre transformation does not depend on whether we couple our external sources to fundamental or composite fields. In practice, though, we have not been able to find a sensible approximation method on this idea.

Nevertheless, we are speculating, so let us assume spontaneous symmetry breakdown does take place in such theories. Then we are led to a remarkable situation: As we have seen, when symmetry breakdown occurs in a fully massless field theory, so does dimensional transmutation; one dimensionless coupling constant disappears, to be replaced by a mass parameter. But spinor electrodynamics (or, for that matter, any theory of gauge fields and fermions based on a simple Lie group) has only *one* dimensionless coupling constant to begin with. Thus, after symmetry breakdown, we are left with *no* free dimensionless coupling constants.

Thus we are led to a program for computing the fine-structure constant. (We emphasize that this is *not* the same as the eigenvalue condition [see (2), above].) To propose such a program is an act of hubris; but we can moderate our ambition and imagine working in a theory in which the gauge group has two simple factors, and thus there are two coupling constants. In this case one would survive, and the fine structure constant would still be a free parameter, but all mass ratios (such as the muon-electron ratio) could be computed in terms of it.

Even this last speculation is certainly very ambitious, and therefore most likely false. Nevertheless, it would be very pleasant to be able to investigate the question in a quantitative way. Thus we feel that the outstanding theoretical challenge posed by our work is to extend our methods to theories without fundamental spinless fields.

ACKNOWLEDGMENTS

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APPENDIX A. MORE LOOPS

1. The Disappearance of Infrared Divergences

In Sec. III we were able to show that the infinite set of one-loop graphs for the effective potential could be summed to give a single integral. We shall now extend this result, and show that the sum of all n-loop graphs can be expressed as a finite sum of n-tuple integrals. Just as in the one-loop case, each of these integrals will be obtained by summing an infinite series of graphs, each of which is infrared-divergent; however, the only relic of the infrared divergences remaining in the sum will be a singularity at the origin of classical field space.

For simplicity, we begin by restricting ourselves to the self-interacting meson theory of Sec. III. For a general graph contributing to the effective potential, we define a type-n vertex to be one with n internal lines (and therefore, $4n$ external lines) attached to it, and denote the number of type-n vertices in the graph by V_n. For a 1PI graph, all vertices are either type-two, type-three, or type-four. The total number of vertices is given by

$$V = V_2 + V_3 + V_4 \ . \qquad \text{(A1)}$$

Since each internal line has two ends, and each end terminates on a vertex, the total number of internal lines is given by

$$2I = 2V_2 + 3V_3 + 4V_4 \ . \qquad \text{(A2)}$$

Thus, the number of loops in the graph is

$$L = I - V + 1$$
$$= \tfrac{1}{2}V_3 + V_4 + 1 \ . \qquad \text{(A3)}$$

Thus, for any fixed L, there exist only a finite number of graphs with L loops and *no* type-two vertices. We shall refer to these as prototype graphs. All other L-loop graphs are generated from the prototype graphs by sticking an arbitrary number of type-two vertices onto the internal lines of the prototype graphs. Figure 7 shows this process for the two possible two-loop prototype graphs.

From this viewpoint, the one-loop graphs of Sec. III are a degenerate case; for them the prototype graph is a graph with no vertices, a simple circle. This is responsible for the peculiar cyclic symmetry of these graphs, which led to the factor of $1/n$ in Eq. (3.2b). For all other cases, the ends of the internal lines in the prototype graph are pinned down to prototype vertices. Therefore, the insertion of type-two vertices does not introduce any new symmetry into the graph, and it is trivial to sum up the result of all such insertions, since we have an independent geometric series for each internal prototype line. Thus we obtain the following *Computational Rule:* To evaluate the sum of all n-loop graphs for n greater than one, compute the sum of all n-loop prototype graphs, but make the following substitution for every internal propagator:

$$\frac{i}{k^2 + i\epsilon} \rightarrow \frac{i}{k^2 - \frac{1}{2}\lambda\varphi_c{}^2 + i\epsilon} \ . \qquad \text{(A4)}$$

This rule may readily be extended to massless scalar electrodynamics; the only difference is

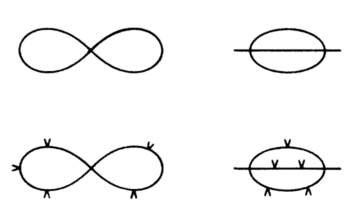

FIG. 7. Above: The only two-loop prototype graphs in self-interacting meson theory. These are $V_4 = 1$, $V_3 = 0$, and $V_4 = 0$, $V_3 = 2$. Below: Two of the infinite series of two-loop graphs obtained from these by inserting type-two vertices on the internal lines.

that for internal photon lines, the substitution is

$$-i\frac{g_{\mu\nu}-k_\mu k_\nu/k^2}{k^2+i\epsilon}\to -i\frac{g_{\mu\nu}-k_\mu k_\nu/k^2}{k^2-e^2\varphi_c{}^2+i\epsilon}\ . \tag{A5}$$

In both cases we see that the infrared divergences in the individual graphs become, in the sum, a singularity at the origin of classical-field space.

Until now, we have only discussed graphs for which all external lines carry zero momentum, but we may sometimes wish to compute graphs for which some of the external momenta are nonzero [for example, to compute $Z(\varphi_c)$]. The computational rule here is the same as before, except that the class of prototype graphs will include some graphs with type-two vertices, those for which one or both of the external momenta attached to the type-two vertices are nonzero. The reason for this is simple: In counting combinations to produce a geometric series, the nonzero momentum serves to distinguish these type-two vertices from the ruck of the others.

2. The Disappearance of Logarithms of Coupling Constants

In Sec. III we showed that, after renormalization, the sum of all the one-loop graphs for the effective potential was proportional to λ^2. We shall now extend this result to many-loop graphs in self-interacting meson theory.

We shall use the analysis in terms of prototype graphs, explained above. In every prototype graph let us define a new momentum variable for each loop,

$$k=\lambda^{1/2}k'\ . \tag{A6}$$

If we express the propagator in terms of the new variable, we find that

$$\frac{1}{k^2-\frac{1}{2}\lambda\varphi_c{}^2}=\frac{\lambda^{-1}}{k'^2-\frac{1}{2}\varphi_c{}^2}\ . \tag{A7}$$

In addition, for every integration momentum, we have

$$d^4k=\lambda^2 d^4k'\ . \tag{A8}$$

Please note that this rescaling of momenta does not affect our renormalization procedure, since our prescription is to subtract graphs at fixed φ_c. Had we followed the more usual renormalization procedure, and subtracted at some fixed Euclidean momentum, the rescaling would change our renormalization conditions, and would not be legitimate.

We can now add up all the powers of the coupling constant associated with a given prototype graph. In addition to the powers explicitly displayed in the preceding two equations, there is, of course, an explicit factor of λ for every vertex. Thus, the contribution to the effective potential is of the form

$$\varphi_c{}^4 f(\varphi_c/M)(\lambda)^{V+2L-I}=\varphi_c{}^4 f(\varphi_c/M)(\lambda)^{L+1}\ , \tag{A9}$$

where M is the renormalization mass and f is a function depending on the graph under consideration.

The situation is somewhat different if we consider graphs in which some external lines carry nonzero momenta. For example, let us consider summing a set of L-loop graphs for which one external line carries a momentum p, another carries a momentum $-p$, and all others carry momentum zero. That is to say, let us consider computing the corrections to the two-point function in a fixed classical field. In this case, to preserve momentum conservation, we must also rescale the external momentum,

$$p=\lambda^{1/2}p'\ . \tag{A10}$$

Then, reasoning as before, we find a result of the form

$$\varphi_c{}^2 g(p'^2/M^2,\varphi_c{}^2/M^2)^{L+1}=\varphi_c{}^2 g(p^2/\lambda M^2,\varphi_c{}^2/M^2)\lambda^{L+1}, \tag{A11}$$

where g is a function which depends on the prototype graph under consideration. Here the logarithms of the coupling constant have not disappeared, but are associated with the logarithmic dependence of the function on p^2.

This has the interesting consequence that if we compute Z by differentiating the above function with respect to p^2 at the origin, we find that the L-loop contribution is proportional to λ^L. This is a natural arrangement of powers from the viewpoint of the renormalization group; it implies that in the L-loop approximation, both terms in the formula for $\overline{\beta}$, Eq. (5.14), are proportional to λ^{L+1}.

It is much more difficult to perform a similar analysis for massless scalar electrodynamics, because there is no rescaling of momenta that will simultaneously eliminate the coupling constants from the denominators of both the meson propagator, Eq. (A4), and the photon propagator, Eq. (A5). Indeed, at this time, we do not know whether the n-loop contributions to the effective potential are simple polynomials in e and λ, or whether they contain logarithms of the coupling constants.

Nevertheless, using an ingenious trick suggested by Politzer,[17] we can investigate the dependence on e of the L-loop contribution to the effective potential, *after* we have determined λ as a function of e by shifting the renormalization point to the minimum of V. Politzer's suggestion is to parametrize the expansion not in terms of λ, as we have been doing, but in terms of another parameter, λ_1, defined by

$$\lambda_1 = \frac{6}{M^3}\frac{\partial V}{\partial \varphi}\bigg|_{\varphi=M} \, . \tag{A12}$$

This is certainly as good (or as bad) a definition of the renormalized coupling constant as our old one; both parameters equal the bare values of λ in zero-loop approximation, which is all that is needed to define an iterative renormalization procedure. In any event, λ (or λ_1) exists only to be eliminated in favor of e at the end of the computation; which parameter we use in the intermediate stages should not be important. The advantage of λ_1 is that it is easy to write in closed form the equation that eliminates it, for, if we shift the renormalization point to a minimum of V, Eq. (A12) becomes

$$\lambda_1 = 0 \, . \tag{A13}$$

Thus, all graphs that involve the quartic-meson self-coupling disappear, and it is easy to construct a rescaling argument to determine the e dependence of the loop expansion. We shall not give the details here, but merely state that the results are the same as those in the previous case, except that λ is replaced by e^2. Thus, the L-loop contribution to the effective potential is proportional to $(e^2)^{L+1}$, etc.

APPENDIX B. MASSIVE SCALAR MESONS

In the body of this paper we restricted ourselves, except for brief digressions, to theories in which the spinless fields were massless, that is to say, in which the renormalization conditions forced $V''(0)$ to vanish. In this appendix we discuss the extension of our techniques to theories in which the scalar mesons have masses, either real or imaginary, that is to say, in which the renormalization conditions force $V''(0)$ to be some fixed nonzero quantity, either positive or negative.

1. Self-Interacting Mesons with Positive Mass

We begin with the self-interacting meson theory of Sec. III, except that we now assume that the meson has a positive mass, μ. The only effect of this is to change the meson propagator from the massless to the massive form. Thus, the one-loop approximation to the effective potential, Eq. (3.3), becomes

$$V = \frac{\lambda}{4!}\varphi_c^{\,4} - \tfrac{1}{2}B\varphi_c^{\,2} - \frac{1}{4!}C\varphi_c^{\,4} + \tfrac{1}{2}\int \frac{d^4k}{(2\pi)^4}\ln\left(1+\frac{\lambda\varphi_c^{\,2}}{2(k^2+\mu^2-i\epsilon)}\right) \, , \tag{B1}$$

where, as before, B and C are renormalization counterterms, and the integral is over Euclidean momentum space. Since in this region $(k^2+\mu^2)$ never vanishes, we can drop the $i\epsilon$. We determine B and C just as in Sec. III, by imposing the renormalization conditions

$$\frac{d^2V}{d\varphi_c^{\,2}}\bigg|_{\varphi_c=0} = \mu^2 \, , \tag{B2}$$

and

$$\frac{d^4V}{d\varphi_c^{\,4}}\bigg|_{\varphi_c=M} = \lambda \, . \tag{B3}$$

The expression for V we obtain in this way is somewhat cumbersome. For simplicity, we will give it explicity only in the regime of interest for the study of the zero-mass limit, $\mu \ll M$:

$$V = \tfrac{1}{2}\mu^2\varphi_c^{\,2} + \frac{\lambda}{4!}\varphi_c^{\,4} + \frac{1}{64\pi^2}\left[(\mu^2+\tfrac{1}{2}\lambda\varphi_c^{\,2})^2\ln\left(1+\frac{\lambda\varphi_c^{\,2}}{2\mu^2}\right) + \tfrac{1}{2}\lambda\mu^2\varphi_c^{\,2} - \tfrac{25}{24}\lambda^2\varphi_c^{\,4} + \tfrac{1}{4}\lambda^2\varphi_c^{\,4}\ln\left(\frac{2\mu^2}{\lambda M^2}\right)\right] . \tag{B4}$$

As the mass goes to zero, this clearly goes smoothly into the corresponding expression for the massless theory, Eq. (3.10).

Of course, since the massive theory is free of infrared divergences, we could choose M to be zero. This is obviously a stupid thing to do if one is studying the zero-mass limit; we could hardly expect a smooth limit as μ goes to zero if we abruptly change the definition of λ at the last moment. Nevertheless, for the reader who might be interested in the massive theory for its own sake, we give the form of V with this renormalization convention:

$$V = \tfrac{1}{2}\mu^2\varphi_c^{\,2} + \frac{\lambda}{4!}\varphi_c^{\,4} + \frac{1}{64\pi^2}\left[(\mu^2+\tfrac{1}{2}\lambda\varphi_c^{\,2})^2\ln\left(1+\frac{\lambda\varphi_c^{\,2}}{2\mu^2}\right) + \tfrac{1}{2}\lambda\mu^2\varphi_c^{\,2} - \tfrac{3}{8}\lambda^2\varphi_c^{\,4}\right] . \tag{B5}$$

For the same hypothetical reader, we state, without proof, that, in the many-meson case, the relevant formulas in Sec. VI, Eqs. (6.2) and (6.3), remain valid in the massive theory if V_0 is interpreted as including quadratic mass terms as well as quartic self-interactions.

2. Self-Interacting Mesons with Imaginary Mass

We now investigate the theory for which μ^2 is negative. In this case, $(k^2+\mu^2)$ can vanish even for Euclidean k; thus we cannot drop the $i\epsilon$ in Eq. (B1), and the effective potential has an imaginary part.[18] This cannot be canceled by the B and C counterterms; these must remain real, to preserve the reality of the Lagrangian.

It is easy to compute this imaginary part:

$$\operatorname{Im} V = -\frac{1}{64\pi}\left[(\mu^2 + \tfrac{1}{2}\lambda\varphi_c^{\,2})^2\,\theta(-\mu^2 - \tfrac{1}{2}\lambda\varphi_c^{\,2}) - \mu^4\right] . \tag{B6}$$

The second term in this equation is just an additive constant; it has no effect on the physics of the system, and can be dropped with impunity. The first term, though, cannot be swept under the rug; it represents a genuine physical effect.

We can gain some heuristic understanding of this effect if we return to the analogy between V and the momentum-space propagator, discussed in Sec. III. In ordinary perturbation theory, the propagator has an imaginary part at those momenta for which the off-mass-shell particle is kinematically unstable. Here, the vacuum itself is kinematically unstable, because of the negative mass term in our initial Lagrangian; thus V develops an imaginary part. We cannot claim credit for discovering this phenomenon; it was first observed in the classic calculation by Euler and Heisenberg[19] of the effective Lagrangian for constant electromagnetic fields in quantum electrodynamics. The Euler-Heisenberg function is real for constant magnetic fields, but imaginary for constant electric fields, because, in the presence of a constant electric field, the vacuum can decay into electron-positron pairs.

Just as in the usual theory of unstable particles, the presence of an imaginary part forces us to modify our mass renormalization condition. Since we can only add real counterterms to the Lagrangian, we can only control the real part of $V''(0)$; thus, for negative μ^2, Eq. (B2) must be replaced by

$$\operatorname{Re} \frac{d^2V}{d\varphi_c^{\,2}}\bigg|_{\varphi_c=0} = \mu^2 . \tag{B7}$$

The physical meaning of μ^2 is now even more remote than before, but it is still a finite quantity which serves to parametrize the theory.

We should observe that the imaginary part of the effective potential does not affect the reality of zero-momentum Green's functions in the physical (asymmetric) theory. For, to lowest order, the vacuum expectation value of φ is given by

$$\mu^2 + \tfrac{1}{6}\lambda\langle\varphi\rangle^2 = 0 . \tag{B8}$$

This is a point at which the θ function in Eq. (B6) vanishes; thus all derivatives of V at this point are real.

3. Massive Scalar Electrodynamics

For scalar electrodynamics,[20] even if the meson has a mass, the photon does not. Thus, diagrams involving photon loops are still infrared divergent, and we must use a nonzero renormalization mass, M, to define the quartic coupling constant, λ. Following Sec. VI, we will assume that λ is of the order of magnitude of e^4, so we can neglect the effects of meson loops compared with those of photon loops. Then, no matter what the value of λ and M, we can find a mass m such that

$$V = \tfrac{1}{2}\mu^2\varphi_c^{\,2} + \frac{3e^4}{64\pi^2}\varphi_c^{\,4}\left(\ln\frac{\varphi_c^{\,2}}{m^2} - \frac{1}{2}\right) . \tag{B9}$$

Note that m is not a renormalization mass but a genuine parameter which, together with e and μ, characterizes the theory; it replaces the redundant pair of variables, λ and M. From Eq. (B9), it is trivial to verify the smooth approach to the zero-mass limit, Eq. (4.9).

Nevertheless, there is still a surprise lurking in this formula: For sufficiently small positive μ^2, V evidently has two local minima, one at the origin and one near m, and its value at the second minimum is evidently lower than its value at the first. Thus, even for positive μ^2, spontaneous symmetry breaking occurs.

This seems to contradict the conventional wisdom stated in Sec. I.[21] Is ordinary real-mass scalar electrodynamics unstable? No, it is not, as we shall now explain.

The point is that the conventional wisdom states that the theory does not suffer spontaneous symmetry breaking if μ^2 is positive *and* λ is positive. As we have said many times, the value of λ depends on our choice of M; in the case at hand, we differentiate Eq. (B9) four times and find

$$\lambda = \frac{9e^4}{8\pi^2}\left(\ln\frac{M^2}{m^2} + \frac{11}{3}\right) . \tag{B10}$$

If we wish to make contact with the conventional description of the massive theory, then we should choose M to be on the order of μ, the only mass in the theory. It would be perverse to do otherwise; we can create spurious contradictions by describing a physical situation in two different formalisms in which we assign the same symbol to two drastically different objects. (Of course, when we are studying the zero-mass limit, we must keep M fixed. Thus M becomes many times larger than μ – but here we are asking a different question.)

Now, if V is to have a second minimum at all (let alone one that is lower than the one at the origin), then it is easy to show that

$$\ln\frac{16\pi^2\mu^2}{3e^4m^2}<-1\ . \tag{B11}$$

For small e (in particular, for e as small as the physical electron charge), this implies that λ is negative if M is on the order of μ.

Thus, properly interpreted, the theory in which we have found a second minimum is not conventional massive scalar electrodynamics at all, but a theory with negative quartic coupling constant. At first glance, we would expect such a theory to have no stable vacuum at all, because, as we move away from the origin, V decreases without bound. However, it is not too surprising to be told that, at least for small λ (on the order of e^4), the effects of radiative corrections dominate those of the quartic self-coupling, and turn V upwards again.

We have argued that this is the natural way to interpret the two minima in Eq. (B8). Even if the reader does not believe this, we think he must admit that it is no less natural than the alternative interpretation (that radiative corrections cause normal massive scalar electrodynamics to suffer spontaneous symmetry breaking), and therefore should be preferred, if only because, when given two otherwise equivalent descriptions of the same physical phenomenon, we should choose the one that does least violence to our intuition.

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[1]We refer here to such familiar problems as infinite cross sections, etc. In fact, it will turn out that these well-known difficulties are somewhat off the point, since they are evidence against the existence of an S matrix but not against the existence of a theory of off-mass-shell Green's functions. We shall show that even the latter type of theory does not exist.

[2]We use a metric with signature (+−−−).

[3]We will return to (and verify in our formalism) this common belief in Appendix B.

[4]For the sophisticated reader, we emphasize that μ^2 is not the bare mass; it is the renormalized mass about the symmetric vacuum. If μ^2 is positive, this is indeed the renormalized mass of the meson. If μ^2 is negative, it possesses no such simple interpretation, but it is still a completely well-defined renormalized quantity which can be used to parametrize the theory. For the unsophisticated reader, we emphasize that these remarks will be explained in greater detail later.

[5]We give here a brief list of some of the standard papers on these phenomena. Spontaneous symmetry breaking: J. Goldstone, Nuovo Cimento 19, 154 (1961); Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961); J. Goldstone, A. Salam, and S. Weinberg, *ibid.* 127, 965 (1962). Higgs phenomenon: F. Englert and R. Brout, Phys. Rev. Letters 13, 321 (1964); P. Higgs, Phys. Letters 12, 132 (1964); G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble, Phys. Rev. Letters 13, 585 (1964); P. Higgs, Phys. Rev. 145, 1156 (1966); T. W. B. Kibble, *ibid.* 155, 1554 (1967). Renormalization: G. 't Hooft, Nucl. Phys. B33, 173 (1971); B35, 167 (1971); B. W. Lee, Phys. Rev. D 5, 823 (1972); B. W. Lee and J. Zinn-Justin, *ibid.* 5, 3121 (1972); 5, 3137 (1972); 5, 3155 (1972).

[6]J. Schwinger, Proc. Natl. Acad. Sci. U. S. 37, 452 (1951); 37, 455 (1951); G. Jona-Lasinio, Nuovo Cimento 34, 1790 (1964). The effective potential was first introduced by Goldstone, Salam, and Weinberg (Ref. 5), but we follow the development of Jona-Lasinio here. A good review of these methods is the contribution of B. Zumino to *Lectures in Elementary Particles and Quantum Field Theory,* edited by S. Deser *et al.* (MIT, Cambridge, Mass., 1970).

[7]Y. Nambu, Phys. Letters 26B, 626 (1968); S. Coleman, J. Wess, and B. Zumino, Phys. Rev. 177, 2238 (1969).

[8]Except for scale invariance, and that is afflicted with anomalies. Nevertheless, as we shall see in Sec. V, we can define masslessness in a general abstract way, by demanding that the theory obey the homogeneous renormalization-group equations rather than the inhomogeneous version of these equations, the Callan-Symanzik equations.

[9]M. Gell-Mann and F. Low, Phys. Rev. 95, 1300 (1954); N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience, New York, 1959). Our notation follows the review of S. Coleman, in Proceedings of the 1971 International Summer School "Ettore Majorana" (Academic, New York, to be published).

[10]For a detailed explanation, see Coleman (Ref. 9).

[11]L. D. Faddeev and V. N. Popov, Phys. Letters 25B, 29 (1967).

[12]Except that the $\frac{1}{2}$ from the reflection symmetry of the diagram is missing; but this is replaced by a $\frac{1}{2}$ that comes from only summing the even terms in the series.

[13]This model was suggested to us by S. L. Glashow and H. Georgi, who collaborated with us on the analysis of its structure. They were attracted to this model because loop effects are important in it even if V contains a mass term, as will be explained in the text.

[14]The discovery that, for certain theories, the scalar-meson part of the Lagrangian inevitably is invariant under a larger symmetry group than the total Lagrangian was made independently by S. Weinberg. More clever than we, he realized, as we did not, that this phenomenon might provide an explanation for the hierarchy of (apparently) broken symmetries found in the real world. [S. Weinberg, Phys. Rev. Letters 29, 1698 (1972).]

[15]S. Weinberg, Phys. Rev. Letters 19, 1264 (1967); A. Salam, in *Elementary Particle Theory: Relativistic Groups and Analyticity* (Nobel Symposium No. 8), edited by N. Svartholm (Wiley, New York, 1969).

[16]The eigenvalue condition was first advanced as a condition for the bare charge in quantum electrodynamics by Gell-Mann and Low (Ref. 9), and by M. Baker, K. Johnson, and R. Willey [Phys. Rev. 136, B1111 (1964)].

It has been advocated as a condition for the physical coupling constants by K. Wilson [Phys. Rev. D 3, 1818 (1971)] and by S. Adler [*ibid.* 5, 3021 (1972)].

[17]H. D. Politzer (private communication).

[18]This was pointed out to us by R. Dashen and D. Gross (private communication).

[19]W. Heisenberg and H. Euler, Z. Physik 98, 714 (1936); J. Schwinger, Phys. Rev. 82, 664 (1951).

[20]The main content of this section is an answer by H. D. Politzer to a question raised by J. D. Bjorken (private communication); our role is reportorial only.

[21]Sec. I, second paragraph.

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Divergence Cancellations in Spontaneously Broken Gauge Theories*

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An extremely simple method is presented to calculate unambiguously higher-order corrections in *the unitary (U) gauge* of theories with spontaneously broken gauge symmetries. Manipulating Feynman integrals in coordinate space, the spurious nonrenormalizable infinities of this gauge are isolated in the form of (contracted) Feynman graphs. Without making reference to any specific global regularization scheme, the *complete* cancellation of these graphs is demonstrated in the cases of fermion-fermion scattering to fourth order in the Abelian model considered by Appelquist and Quinn and for the similar neutrino scattering in Weinberg's $SU(2)_L \times Y$ model. The reason for such complete cancellation is seen to be a consequence of the algebraic structure of the equal-time commutators among currents, their divergences, and various fields. This structure, of course, is dictated by the original gauge symmetry. As a check on our methods, the weak muon anomaly in Weinberg's model is calculated, and agreement is found with the (gauge-invariant) results of other authors.

I. INTRODUCTION

One of the most interesting recent developments of field theory is the discovery of models of "quasirenormalizable" type. Weinberg[1] presented the first such realistic model, in which he proposed to unify the weak and electromagnetic interactions of leptons through the spontaneous breakdown of symmetry in a gauge-invariant Lagrangian via the Higgs phenomenon.[2] By now a large number of models of this type have been proposed which also include hadronic interactions.[3]

A common feature of all these theories is the fact that, to a given order of perturbation theory, individual Feynman diagrams contributing to a specific process contain nonrenormalizable divergences. We shall call these infinities "spurious" because they happen to cancel when all individual diagrams are added. Cancellations of this type have been shown to occur in the Weinberg theory[1,4] and in a simplified Abelian model[5] by means of a simple cutoff prescription to regulate the integrals involved.

It is well known for renormalizable theories of the usual type that different arbitrary regularizations yield different *finite* additions to the divergent parts of Feynman graphs.[6] This is not serious because the infinite *together* with such (ambiguous) finite parts are absorbed into mass and charge renormalization anyhow. In quasirenormalizable theories some of the infinities – the spurious ones – cancel against infinities in other graphs. Here finite, regularization-dependent terms can be left over. Such an ambiguity has already been encountered in the early calculations of the anomalous magnetic moment of the muon in Weinberg's theory.[7]

We have been referring so far to the "U-gauge" formulation of the theory only. In this gauge the fields are redefined so as to eliminate superfluous scalar bosons. Just because of the spurious-divergence problem mentioned above, the proof of renormalizability for these theories has not been given in the U gauge. In a series of fundamental papers 't Hooft[8] and Lee and Zinn-Justin[9] have given the proof in the so-called "R gauge," where