QFT – assignment 1: EM Fields and Ladder Operators

1. Show that the radiation field is transverse, $\nabla \cdot \vec{A} = 0$ and obeys the wave equation $\nabla^2 \vec{A} - \frac{1}{c^2}\partial_t^2 \vec{A} = 0$. Your starting point should be the mode expansion of the quantum field.

2. Prove that $H = \sum_{k\alpha} \hbar\omega N_{k\alpha}$ and $\vec{P} = \sum_{k\alpha} \hbar\vec{k} N_{k\alpha}$. Use the mode expansion of the quantum field, $H = 1/2\int(E^2 + B^2)d^3x$, and $\vec{P} = 1/c\int E \times Bd^3x$.

3. **gauge invariance of a nonrelativistic particle coupled to photons**
Prove that if $H\psi = i\hbar\dot{\psi}$ where $H = \frac{1}{2m}(\vec{p} - e/c\vec{A})^2 + e\phi$ then a solution to the gauge transformed Schrödinger equation (obtained by setting $\vec{A} \to \vec{A} + \nabla\Lambda$, $\phi \to \phi - \dot{\Lambda}$) is given by

$$\psi_\Lambda = \mathrm{e}^{\frac{ie}{\hbar c}\Lambda(x)}\psi \tag{1}$$

Ie, $H_\Lambda\psi_\Lambda = i\hbar\dot{\psi}_\Lambda$.

Thus the spectrum is preserved under a gauge transformation, and the imposition of a gauge transformation requires the simultaneous rotation of the phase of the wavefunction.

Note that this result implies that the general solution to Schrödinger's equation in the presence of a vector potential can be written as

$$\psi(x) = \psi^{(0)}(x)\exp[\frac{ie}{\hbar c}\int^{\vec{x}} \vec{A}(\vec{y}) \cdot d\vec{s}(\vec{y})] \tag{2}$$

where $\psi^{(0)}$ satisfies the Schrödinger equation with $\vec{A} = 0$.

4. **coherent states** [Mandl and Shaw, 1.1]
A coherent state is a minimum uncertainty wavepacket, and thus represents the 'most classical' wavefunction that can be formed. Let

$$|c\rangle = \mathrm{e}^{-|c|^2/2}\sum_n \frac{c^n}{\sqrt{n!}}|n\rangle \tag{3}$$

where c is a complex number and $|n\rangle = \frac{(a_{k\alpha}^\dagger)^n}{\sqrt{n!}}|0\rangle$. Prove that
(i) $|c\rangle$ is normalised
(ii) $a_{k\alpha}|c\rangle = c|c\rangle$
(iii) $n = \langle c|\hat{N}|c\rangle = |c|^2$
(iv) $\langle c|\vec{E}(\vec{x},t)|c\rangle = -\sum_k \sqrt{\frac{2\hbar\omega}{V}}\vec{\epsilon}(k,\alpha)|c|\sin(\vec{k}\cdot\vec{x} - \omega t + \arg(c))$. Use linear polarisation vectors.