QFT assignment 7: Renormalisation Group, Two-Loops, and Coupling Flow

1. Consider the Callan-Symanzik equation

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - n\gamma(g)\right]\mathcal{M}^{(n)}(\{p\}; g, \mu) = 0$$

Notice that $\mathcal{M}$ depends on many momenta so it is not convenient to use Euler's theorem to trade μ for p in this case. But one can imagine rescaling all the momenta by a factor ξ and we can study ξ-dependence of $\mathcal{M}$. Assume that $\mathcal{M}^{(n)}$ scales as E^{4-n}.

(i) Show that in this case the Callan-Symanzik equation becomes

$$\left[4 - n - \xi\frac{\partial}{\partial\xi} + \beta(g)\frac{\partial}{\partial g} - n\gamma(g)\right]\mathcal{M}^{(n)}(\{\xi p\}; g, \mu) = 0$$

The solution to this equation is

$$\mathcal{M}^{(n)}(\{\xi p\}, g, \mu) = \xi^{4-n}\exp\left[-n\int^{\xi}\frac{dx}{x}\gamma(\bar{g}(x, g))\right]\mathcal{M}^{(n)}(\{p\}, \bar{g}, \mu)$$

where

$$\xi\frac{\partial}{\partial\xi}\bar{g}(\xi, g) = \beta(\bar{g})$$

and $\bar{g}(1, g) = g$.

(ii) Show that

$$\left(\xi\frac{\partial}{\partial\xi} - \beta(g)\frac{\partial}{\partial g}\right)\bar{g}(\xi, g) = 0.$$

(iii) Use the result (ii) to establish (i) by explicitly differentiating.

2. [Peskin & Schroeder 10.3] Compute the setting sun diagram in massless ϕ^4 theory. Verify that the loop counterterm diagram is zero for this theory.

3. [Peskin & Schroeder 12.3] Consider the field theory defined by

$$\mathcal{L} = \frac{1}{2}\left((\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2\right) - \frac{\lambda}{4}(\phi_1^4 + \phi_2^4) - \frac{2\rho}{4}\phi_1^2\phi_2^2.$$

Notice that at $\rho = \lambda$ the theory has an $O(2)$ invariance under rotations of (ϕ_1, ϕ_2).

(i) Compute the beta functions for ρ and λ. Work in four dimensions.

(ii) Write the renormalisation group equation for ρ/λ. Show that if $\rho/\lambda < 3$ at a renormalisation scale M, then this ratio flows towards the point $\rho = \lambda$ in the IR. Comment on your result.

(iii) Obtain the beta functions in $4-\epsilon$ dimensions. Show that there are fixed points at $\rho/\lambda = 0, 1, 3$. Which of these is the most stable? Sketch the pattern of coupling constant flows.