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POINT CANONICAL TRANSFORMATIONS IN THE PATH INTEGRAL *

J.-L. GERVAIS
Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, 24 rue Lhomond, 75005 Paris

A. JEVICKI
Department of Physics, The City College of The City University of New York, New York, New York 10031

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We discuss point canonical transformations using the path integral quantization procedure. First, employing the Feynman diagram technique we demonstrate how a formal treatment of point transformations leads to erroneous results. Then we present the correct treatment of this problem using a more precise definition of path integrals. We show how this more careful treatment leads to additional potential terms in the action as compared to the formal treatment. We also demonstrate the equivalence of our results with the corresponding operator method discussion. Furthermore, we investigate the consequences of these results on the path integral collective coordinate method. Giving the improved treatment on the one-soliton sector example we establish the path integral collective coordinate method of the same level of rigor as the parallel operator approaches.

1. Introduction

In the last few years the path integral method has been used for the quantization of various field theories. It is a very effective method for deriving the Feynman rules of, for example, non-linear chiral theories, Yang-Mills gauge theories and gravity. Furthermore, the possibility of performing field transformations in this method by simply changing integration variables in the path integral allows for a very elegant derivation of Ward-Takahashi identities and discussion of gauge independence in these theories.

Recently, the path integral quantization method has proven to be extremely useful in deriving non-perturbative techniques in field theory. In collaboration with Sakita we developed a path integral collective coordinate method [1, 2] to study extended particles in quantum field theory. Similar approaches were given also by

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other authors [3–5] and in ref. [2] we presented a general formation of this method using the phase space path integral quantization procedure.

The main idea of the collective coordinate method consists in introducing new canonical variables representing the symmetry degrees of freedom. This then guarantees that the symmetries of our field theory are indeed maintained when perturbing about the classical extended particle solutions. In general, collective coordinates are introduced into the action of the path integral expression for the transition amplitude by a canonical transformation generated by the symmetry generators. If the generator is quadratic in fields this is a point transformation but otherwise it is a more general canonical transformation.

It is not known at present how to perform general canonical transformations in quantum theory and this limitation is valid for path integral formalism as well. But it is generally believed that point canonical transformations can be performed by simply changing integration variables in the path integral. It is this question on which we elaborate in the first part of this paper.

First, in sect. 2 we demonstrate, using the Feynman diagram technique, that the above statement about point transformations appears to be incorrect. Namely, we show in a simple example that a formal change of variables in the path integral leads to erroneous results.

Let us emphasize in the very beginning that the above example is not given with the intention of questioning the validity of the path integral method. On the contrary, we like to show that the method is perfectly consistent and rigorous, but that a little more care is needed when handling path integrals. Therefore, in sect. 3 we present a correct discussion of point transformations in path integrals. We show how this more careful treatment leads to additional potential terms in the action of the path integral as compared with the formal treatment. We derive a general expression for this additional potential term in case of an arbitrary point transformation.

The result derived in sect. 3 may have important consequences in connection with non-linear chiral theories and quantum gravity. We don't discuss these questions in this work but only concentrate on the consequences concerning the path integral collective coordinate method. For definiteness, we consider the one-soliton sector case.

In deriving the one-soliton sect or perturbation expansion in path integral method [1–4], a point canonical transformation was performed formally by changing integration variables. In parallel operator formulation [6] it was observed that there is an ordering problem and that it is not clear how it can be solved in path integral approach. Now, using the results of sect. 3 we elaborate on this question giving the improved treatment in sect. 4.

Finally we demonstrate in the appendix that the results we have obtained in the path integral approach are in agreement with the operator method derivation.

The purpose of this work is to show that with careful understanding of path integrals, one can discuss in this method such delicate questions as the ordering of factors when changing integration variables and consequently derive the correct

Feynman rules for theories with an ordering problem. This then establishes the general path integral collective coordinate method at the same level of rigor as the parallel operator approach [6, 7].

2. Canonical transformations and Feynman diagrams

In this section we will make some observations concerning the problem of performing canonical transformation in functional method. We restrict ourselves to point canonical transformations only.

In the literature on functional methods [8, 9] one often finds the statement that point transformations can be performed by simply changing integration variables in the functional integrals. Since there exists a direct correspondence between functional integrals and Feynman diagrams, all manipulations carried out with functional integrals can be explicitly checked using the diagram technique. Thus, in this section we will use the Feynman diagram technique to question the statement about point transformations made above. Namely, we will demonstrate by explicit calculations on a simple example that in general it is not correct to perform point canonical transformations by just making naive changes of variables in the functional integral. The fact is that after this formal change of variables we end up with a quantum theory entirely different from the original one. These two theories are identical only at the tree and one-loop level. The difference between them appears already at the two-loop level and that is what we are going to demonstrate in what follows.

Although point canonical transformations can be discussed in Lagrangian functional formalism we prefer to use the phase space path integral method throughout this section. Let us consider a simple example of N free harmonic oscillators coupled to external sources

$$H_J(p, q) = \sum_{a=1}^{n} (\tfrac{1}{2}p_a^2 + \tfrac{1}{2}\omega^2 q_a^2 - J_a q_a) \ . \tag{2.1}$$

The generating functional is defined as usual and is represented in path integral form by

$$Z(J) = \int \prod_{a=1}^{n} Dp_a Dq_a \exp\left\{\frac{i}{\hbar}\int dt\, [p \cdot \dot{q} - H_J(p, q)]\right\} \ . \tag{2.2}$$

With the small negative imaginary part added to ω^2 we obtain the answer for this path integral as

$$Z(J) = \exp\{-\tfrac{1}{2} \sum_{a, b} \int dt\, dt'\, J_a(t)\, \Delta_F^{ab}(t - t')\, J_b(t')\} \ , \tag{2.3}$$

where we have the Feynman propagators

$$\Delta_{\mathrm{F}}^{ab}(t-t') = \delta_{ab}\,\Delta_{\mathrm{F}}(t-t'), \qquad \Delta_{\mathrm{F}}(t-t') = \int \frac{\mathrm{d}\nu}{2\pi}\, \mathrm{e}^{i\nu(t-t')} \frac{i}{\nu^2-\omega^2+i\epsilon} \, . \tag{2.4}$$

Now, we perform a general point transformation from the old variables p_a, q_a to new canonical variables P_i, Q_i as follows:

$$q_a(t) = F^a(Q(t)), \qquad p_a(t) = F^a_{,i}(Q(t))g^{ij}(Q(t))P_j(t) \, . \tag{2.5}$$

We denote

$$F^a_{,i}(Q) = \frac{\partial F^a(Q)}{\partial Q_i} \qquad i = 1, 2, ..., n \, ,$$

$$g_{ij}(Q) = \sum_{a=1}^{n} F^a_{,i}(Q)F^a_{,i}(Q) \, , \tag{2.6}$$

and $g^{ij}(Q)$ represents the inverse matrix of $g_{ij}(Q)$. The Jacobian of this transformation is one and thus by simply changing integration variables in the path integral expression for the generating functional, we arrive at the new path integral

$$\bar{Z}(J) = \int \prod_{i=1}^{n} DP_i DQ_i \exp\left\{\frac{i}{\hbar}\int \mathrm{d}t [P\cdot \dot{Q} - \bar{H}_J(P, Q)]\right\}, \tag{2.7}$$

with the new Hamiltonian $\bar{H}_J$ given by

$$\bar{H}_J(P, Q) = H_J(p(P, Q), q(Q))$$

$$= \sum_{i,j=1}^{n} \tfrac{1}{2} g^{ij}(Q)P_iP_j + \sum_{a=1}^{n}(\tfrac{1}{2}\omega^2 F^a(Q)F^a(Q) - J_a F^a(Q)) \, . \tag{2.8}$$

We denoted this path integral expression for the generating functional by a different symbol $\bar{Z}(J)$ since it is not clear whether the naive change of variables is a correct step or not. Indeed, in what follows, we are going to demonstrate that contrary to naive expectations $\bar{Z}(J)$ is not equal to $Z(J)$.

Since our Hamiltonian expressed in terms of new canonical variables P_i and Q_i contains in general complicated interaction terms the only way we can calculate the new generating functional is by perturbation theory. To find the Feynman rules of this perturbation expansion we write as usual:

$$\bar{Z}(J) = \exp\left\{-\frac{i}{\hbar}\int \mathrm{d}t\, \bar{H}_{\mathrm{int}}\left(\frac{1}{i}\frac{\delta}{\delta K(t)}, \frac{1}{i}\frac{\delta}{\delta J(t)}\right)\right\} Z_0(J, K)\Big|_{K=0} \, , \tag{2.9}$$

where the free generating function is given by

$$Z_0(J,K) = \int \prod_{i=1}^{n} DP_i DQ_i \exp\left\{\frac{i}{\hbar}\int \mathrm{d}t [P\cdot\dot{Q} - \tfrac{1}{2}(P\cdot P + \omega^2 Q\cdot Q) + J\cdot Q + K\cdot P]\right\} . \tag{2.10}$$

We introduced additional sources K_i coupled to momentum variables P_i since the interaction Hamiltonian contains derivative interactions. The free generating functional $Z_0(J,K)$ is easily found to be

$$Z_0(J,K) = \exp\{-\tfrac{1}{2}\sum_{i,j=1}^{n}\int \mathrm{d}t\,\mathrm{d}t'(J_i(t)-\dot{K}_i(t))\,\Delta_F^{ij}(t-t')(J_j(t')-\dot{K}_j(t')) - \tfrac{1}{2}i\sum_{i=1}^{n}\int \mathrm{d}t\, K_i(t)K_i(t)\} , \tag{2.11}$$

giving propagators

$$\langle 0|T\{\hat{Q}_i(t), \hat{Q}_j(t')\}|0\rangle = \delta_{ij}\Delta_F(t-t') ,$$
$$\langle 0|T\{\hat{P}_i(t), \hat{Q}_j(t')\}|0\rangle = \delta_{ij}\frac{\mathrm{d}}{\mathrm{d}t}\Delta_F(t-t') ,$$
$$\langle 0|T\{\hat{P}_i(t), \hat{P}_j(t')\}|0\rangle = \delta_{ij}\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}t'}\Delta_F(t-t') - i\delta(t-t')\right] ,$$
$$i,j = 1,2,\ldots,n , \tag{2.12}$$

represented graphically in fig. 1. The Feynman rules are completed by an infinite set of vertices obtained by expanding $F^a(Q)$ and $g^{ij}(Q)$ in powers of Q_i,

$$F^a(Q) = F^a + F^a_{,i}Q_i + \frac{1}{2!}F^a_{,ij}Q_iQ_j + \frac{1}{3!}F^a_{,ijl}Q_iQ_jQ_l + \ldots ,$$
$$g^{ij}(Q) = g^{ij} + g^{ij}_{,l}Q_l + \frac{1}{2!}g^{ij}_{,lm}Q_lQ_m + \ldots . \tag{2.13}$$

For simplicity we choose $F^a = 0$ and $F^a_{,ji} = \delta_{ji}$. The Hamiltonian from which we read of the vertices is now

$$\bar{H}_J(P,Q) = \tfrac{1}{2}\sum_{i,j} P_iP_j(\delta_{ij} + g^{ij}_{,l}Q_l + \frac{1}{2!}g^{ij}_{,lm}Q_lQ_m + \ldots)$$
$$+ \tfrac{1}{2}\omega^2[Q\cdot Q + F^l_{,ij}Q_iQ_jQ_l + (\tfrac{1}{3}F^l_{,ijm} + \tfrac{1}{4}F^a_{,ij}F^a_{,lm})Q_iQ_jQ_lQ_m + \ldots]$$

$$-\sum_{a=1}^{n} J_a\left(Q_a + \frac{1}{2!}F^a_{,ij}Q_iQ_j + \dots\right). \tag{2.14}$$

The first few in this series of vertices are represented graphically in fig. 2.

Now we have a loop expansion for $\bar{Z}(J)$. One can easily see that due to cancellations of graphs when calculating $\bar{Z}(J)$ at tree and one-loop level the result is equal to $Z(J)$. As an example of such cancellations consider the one-loop tadpole diagrams shown in figs. 3a, 3b and 3c. Their respective contributions are

$$(a) = -\frac{i}{4\omega}\left(F^l_{,ij} + 2F^i_{,li}\right)\int \mathrm{d}t\, J_l(t)\,,$$

$$(b) = \frac{i}{2\omega}F^i_{,il}\int \mathrm{d}t\, J_l(t)\,,$$

$$(c) = \frac{i}{4\omega}F^l_{,ii}\int \mathrm{d}t\, J_l(t)\,, \tag{2.15}$$

where we used the identity

$$g^{ij}_{,l}(Q) = -g^{ii'}(Q)g^{jj'}(Q)g_{i'j',l}(Q)\,. \tag{2.16}$$

Obviously the sum of these three terms is zero. For a general discussion, the reader is referred to ref. [9].

But, unfortunately these cancellations do not persist already at the next two-loop level. Consider, for example, all the two-loop bubble diagrams shown in figs.

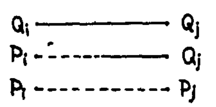

Fig. 1. Propagators.

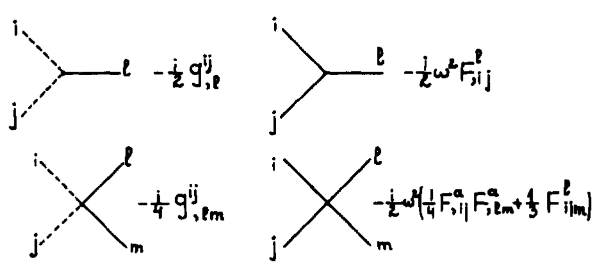

Fig. 2. Vertices.

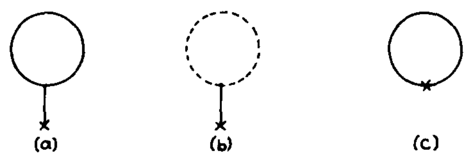

Fig. 3. One-loop tadpole diagrams.

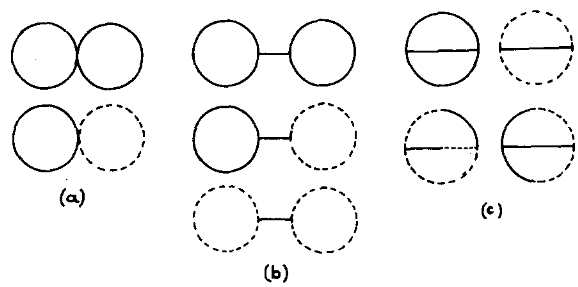

Fig. 4. Two-loop bubble diagrams.

4a, 4b and 4c. After some calculation using the integrals

$$\int \mathrm{d}t\, \Delta_{\mathrm{F}}^{3}(t) = -\tfrac{1}{12}\,\frac{i}{\omega^{4}}\,,$$

$$\int \mathrm{d}t\, \dot{\Delta}_{\mathrm{F}}^{2}(t)\, \Delta_{\mathrm{F}}(t) = \tfrac{1}{12}\,\frac{i}{\omega^{2}}\,, \tag{2.17}$$

we find the contributions of these graphs:

$$(a) = \tfrac{1}{8}\, i\hbar\left(-g_{ij,l}\,g_{ij,l} - \tfrac{1}{4} F^{a}_{,ii} F^{a}_{,jj} + \tfrac{1}{2} F^{a}_{,ij} F^{a}_{,ij}\right)\,,$$

$$(b) = \tfrac{1}{8}\, i\hbar\, \tfrac{1}{4} F^{a}_{,ii} F^{a}_{,jj}\,,$$

$$(c) = \tfrac{3}{8}\, i\hbar\left(\tfrac{1}{2} F^{a}_{,ij} F^{a}_{,ij} + F^{l}_{,ij} F^{i}_{,jl}\right)\,. \tag{2.18}$$

The sum of these three terms does not give zero but rather

$$(a) + (b) + (c) = \tfrac{1}{8}\, i\hbar\, F^{l}_{,ij} F^{i}_{,jl} \neq 0\,. \tag{2.19}$$

This explicit calculation shows that $\bar{Z}(J)$ is indeed different from $Z(J)$. So the new path integral (2.7) with the Hamiltonian $\bar{H}_J(P, Q)$ does not define the same theory as the original one.

We now like to emphasize that the discussion presented above is not given in intention to question the validity of the path integral method. Namely, the result of this section just means that it is incorrect to naively perform point transformations in path integral and that more care is needed when doing so.

In the next section we will present a careful treatment of this problem and it will be shown that it leads to additional potential terms $\Delta V(Q)$ in the action of the new path integral (2.7) meaning that it is the modified Hamiltonian

$$\tilde{H}'_J(P, Q) = H_J(p(P, Q), q(Q)) + \Delta V(Q)$$

which describes the same quantum theory as the original one.

3. Point canonical transformations in the path integral method

The path integral method is often considered as of heuristic value only, with the understanding that all results derived in this approach are to be checked by parallel calculations in the operator formalism. There is also a belief that path integrals can safely be used only through their correspondence with the diagram technique, so that manipulations with path integrals just represent manipulations with Feynman diagrams.

We do not accept such limited definitions of the path integral method, especially in view of the fact that this method proves to be extremely useful in developing non-perturbative techniques. Although as of yet there exists no rigorous mathematical definition of integration over paths, there is still a precise enough formulation of the method which allows us to derive conclusions as rigorously as with the operator formalism. This is the original definition introduced by Feynman which consists in defining the path integral as a limit of finite dimensional integrations. It is discussed in detail in ref. [10] with the special emphasis on connection with the operator ordering problem.

In this section we make use of this precise definition in order to demonstrate how it allows for a rigorous treatment of point canonical transformations in the path integral method.

We consider a general theory described by the Lagrnagian

$$L(q, \dot{q}) = \tfrac{1}{2} \sum_{a=1}^{n} \dot{q}_a^2 - V(q) \ . \tag{3.1}$$

The kernel to go from some initial point $q' = (q'_1, q'_2, \ldots, q'_n)$ at time t' to some final point $q'' = (q''_1, q''_2, \ldots, q''_n)$ at time t'' is given by the following path integral:

$$K(q'', t''; q', t') = \int_{q(t')=q'}^{q(t'')=q''} \prod_{a=1}^{n} Dq_a \exp\left\{\frac{i}{\hbar} A[q]\right\} \tag{3.2}$$

the right-hand side being defined as the limit of finite dimensional integrations. Namely, we subdivide the time $t'' - t'$ into N equal intervals denoting $t_k = t' + \epsilon k$ and $q_a(k) \equiv q_a(t_k)$. Then the precise expression for this path integral reads

$$\lim_{N\to\infty} \int \prod_{a=1}^{n} \prod_{k=1}^{N-1} \frac{\mathrm{d}q_a(k)}{(2\pi i\hbar\epsilon)^{1/2}} \prod_{k=0}^{N-1} \exp \frac{i}{\hbar} A[q(k+1), q(k)] \quad , \tag{3.3}$$

with $q_a(0) = q_a'$ and $q_a(N) = q_a''$. In the exponent we have the short time action

$$A[q(k+1), q(k)] = \frac{1}{2\epsilon} \sum_{a=1}^{n} (q_a(k+1) - q_a(k))^2 - \epsilon V(q(k)) \ . \tag{3.4}$$

We now perform a general point canonical transformation $q_a(t) = F^a(Q(t))$ to new variables $Q = (Q_1, Q_2, ..., Q_n)$. This means that in the finite dimensional integral we have to change variables substituting $q_a(k) = F^a(Q(k))$. Then the new integration measure is given by

$$\prod_{a=1}^{n} \mathrm{d}q_a(k) = [\det g_{ij}(Q(k))]^{1/2} \prod_{i=1}^{n} \mathrm{d}Q_i(k) \ , \tag{3.5}$$

and the new short time action is simply

$$\begin{aligned} \widetilde{A}[Q(k+1), Q(k)] &= A[F(Q(k+1)), F(Q(k))] \\ &= \frac{1}{2\epsilon} \sum_{a=1}^{n} (F^a(Q(k+1)) - F^a(Q(k)))^2 - \epsilon V(F(Q(k))) \ . \end{aligned} \tag{3.6}$$

At this point it is tempting to use the expansion

$$\begin{aligned} F^a(Q(k+1)) - F^a(Q(k)) &= F^a_{,i}(Q(k)) \Delta Q_i(k) \\ &+ \frac{1}{2!} F^a_{,ij}(Q(k)) \Delta Q_i(k) \Delta Q_j(k) + \dots , \end{aligned} \tag{3.7}$$

and approximate the short time action (3.6) as

$$\widetilde{A}[Q(k+1), Q(k)] \approx \frac{1}{2\epsilon} \sum_{i,j=1}^{n} g_{ij}(Q(k)) \Delta Q_i(k) \Delta Q_j(k) - \epsilon V(F(Q(k))) \ , \tag{3.8}$$

thus dropping the higher order terms in $\Delta Q_i(k) = Q_i(k+1) - Q_i(k)$.

But the above approximation appears to be incorrect and would lead us to erroneous results. In fact, it is equivalent to the naive change of variables discussed in sect. 2. The mistake which would be made by making this approximation has its origin in the observation that higher order terms in the short time action of the form $(\Delta Q)^3/\epsilon$ and $(\Delta Q)^4/\epsilon$ still contribute to the path integral and thus cannot be neglected..Namely, due to the stochastic nature of path integrals we have that $\Delta Q \neq O(\epsilon)$ but rather $(\Delta Q)^2 = O(\epsilon)$. This observation is due originally to Edwards and Gulyaev who investigated the change from Cartesian to polar coordinates in the path integral formalism [11].

So in order to correctly approximate the short time action we have to keep all terms effectively of $O(\epsilon)$. We choose to expand the functions $F^a(Q(k+1))$ and $F^a(Q(k))$ about the midpoint $2\bar{Q}_i(k) = Q_i(k+1) + Q_i(k)$ and thus it is the symmetric ("midpoint") definition of the path integral which we adopt. Then keeping terms of up to the fourth order in ΔQ we obtain the following approximation for the short time action:

$$\bar{A}[Q(k+1), Q(k)] = \frac{1}{2\epsilon}(g_{ij}(\bar{Q}(k))\,\Delta Q_i(k)\,\Delta Q_j(k)$$
$$+ \tfrac{1}{12}F^a_{,i}(\bar{Q}(k))\,F^a_{,jlm}(\bar{Q}(k))\,\Delta Q_i(k)\Delta Q_j(k)\Delta Q_l(k)\Delta Q_m(k)) - \epsilon V(F(\bar{Q}(k))). \tag{3.9}$$

We now have the correct path integral expression for the kernel in terms of new variables Q_i

$$[\det g_{ij}(Q'')\det g_{ij}(Q')]^{-1/4}\lim_{N\to\infty}\int\prod_{i=1}^{n}\prod_{k=1}^{N-1}\frac{\mathrm{d}Q_i(k)}{(2\pi i\epsilon\hbar)^{1/2}}$$
$$\times\prod_{k=0}^{N-1} J(Q(k+1), Q(k))\exp\left\{\frac{i}{\hbar}\bar{A}[Q(k+1), Q(k)]\right\}, \tag{3.10}$$

with the Jacobian given by

$$J(Q(k+1), Q(k)) = [\det g_{ij}(Q(k+1))\det g_{ij}(Q(k))]^{1/4}\ . \tag{3.11}$$

In order to have the symmetric path integral we still need to expand the Jacobian about $\bar{Q}_i(k)$ keeping quadratic terms in $\Delta Q_i(k)$ since they still contribute, being of $O(\epsilon)$. Using the identity

$$\det(A+B) = \det A\,\det(1 + A^{-1}B) = \det A\,\{1$$
$$+\mathrm{Tr}(A^{-1}B) + \tfrac{1}{2}(\mathrm{Tr}\,A^{-1}B)^2 - \tfrac{1}{2}\mathrm{Tr}\,(A^{-1}B)^2 + \dots\}, \tag{3.12}$$

we obtain after some calculation

$$J(Q(k+1), Q(k)) \approx [\det g_{ij}(\bar{Q}(k))]^{1/2}[1+\tfrac{1}{16}(g^{ij}(\bar{Q}(k))\, g_{ij,lm}(\bar{Q}(k)) + g^{ij}_{,l}(\bar{Q}(k))\, g_{ij,m}(\bar{Q}(k)))\,\Delta Q_l(k)\Delta Q_m(k)] \ . \tag{3.13}$$

Although now we have a correct path integral expression for the kernel in terms of the new variables Q_i it is not very useful since in the action of (3.13) there are additional terms of the form $(\Delta Q)^4/\epsilon$ and also in the Jacobian terms of the form $(\Delta Q)^2$. One would prefer to eliminate these terms in favor of a potential like term of the form $\epsilon V(Q)$. That can be done and a detailed discussion of this problem was already given by McLaughlin and Schulman in connection with the path integral quantization in curved spaces. There the same problem appears [12]. We will not repat their arguments here but only give a short derivation of our results.

We start by approximating the exponential in (3.10) by

$$\exp\left\{\frac{i}{2\hbar\epsilon}\sum_{i,j} g_{ij}(\bar{Q}(k))\,\Delta Q_i\Delta Q_j - \epsilon V(F(\bar{Q}(k)))\right\} \times \left(1+\frac{i}{24\hbar\epsilon}F^a_{,i}(\bar{Q})\,F^a_{,jlm}(\bar{Q})\,\Delta Q_i\Delta Q_j\Delta Q_l\Delta Q_m\right), \tag{3.14}$$

so that the effective Jacobian has the following form:

$$g(\bar{Q})^{1/2}\{1+\tfrac{1}{16}(g^{ij}(\bar{Q})\,g_{ij,lm}(\bar{Q})+g^{ij}_{,l}(\bar{Q}))\,\Delta Q_l\Delta Q_m + \frac{i}{24\hbar\epsilon}F^a_{,i}(\bar{Q})F^a_{,jlm}(\bar{Q})\,\Delta Q_i\Delta Q_j\Delta Q_l\Delta Q_m\} \ . \tag{3.15}$$

Then making use of the following integrals

$$\int\prod_{i=1}^{n} \mathrm{d}x_i \exp\left[\frac{i}{2\hbar\epsilon}g_{ij}x_ix_j\right]x_\alpha x_\beta = (2\pi i\epsilon\hbar)^{n/2}g^{-1/2}(i\hbar\epsilon)g^{\alpha\beta} \ ,$$

$$\int\prod_{i=1}^{n} \mathrm{d}x_i \exp\left[\frac{i}{2\hbar\epsilon}g_{ij}x_ix_j\right]x_\alpha x_\beta x_\gamma x_\delta = (2\pi i\epsilon\hbar)^{n/2}g^{-1/2}(i\hbar\epsilon)^2 \times (g^{\alpha\beta}g^{\gamma\delta}+g^{\alpha\gamma}g^{\beta\delta}+g^{\alpha\delta}g^{\beta\gamma}) \ , \tag{3.16}$$

we substitute the additional terms in the Jacobian (3.15) by potential like terms of the form

$$\epsilon\Delta V(\bar{Q}) = \epsilon\hbar^2\{-\tfrac{1}{16}(g^{ij}(\bar{Q})g_{ij,lm}(\bar{Q})+g^{ij}_{,l}(\bar{Q})\,g_{ij,m}(\bar{Q}))\,g^{lm}(\bar{Q}) + \tfrac{1}{24}F^a_{,i}(\bar{Q})F^a_{,ilm}(\bar{Q})(g^{ij}(\bar{Q})\,g^{lm}(\bar{Q})+g^{il}(\bar{Q})\,g^{jm}(\bar{Q})+g^{im}(\bar{Q})\,g^{jl}(\bar{Q}))\} \ . \tag{3.17}$$

Now using the identity

$$\frac{1}{g(Q)}\, g_{,l}(Q) = g^{ij}(Q)\, g_{ij,l}(Q) \tag{3.18}$$

we simplify this expression

$$\Delta V(\bar{Q}) = \tfrac{1}{8}\hbar\{F^a_{,i}(\bar{Q})\, F^a_{,jlm}(\bar{Q})\, g^{ij}(\bar{Q})\, g^{lm}(\bar{Q})$$
$$-\tfrac{1}{2}\left(\frac{1}{g(\bar{Q})}\, g_{,l}(\bar{Q})\right)_{,m} g^{lm}(\bar{Q}) \quad . \tag{3.19}$$

Next introducing the Christoffel symbols defined by

$$F^a_{,ij}(Q) = \Gamma^l_{ij}(Q)\, F^a_{,l}(Q) \ , \tag{3.20}$$

we find

$$F^a_{,i}(Q)\, F^a_{,jlm}(Q)\, g^{ij}(Q)\, g^{lm}(Q) = \Gamma^i_{il,m}(Q)\, g^{lm}(Q) + g^{lm}(Q)\, \Gamma^j_{il}(Q)\, \Gamma^i_{jm}(Q) \ , \tag{3.21}$$

which together with the identity

$$\Gamma^j_{jl,m}(Q) = \tfrac{1}{2}\left(\frac{1}{g(Q)}\, g_{,l}(Q)\right)_{,m} \tag{3.22}$$

leads us to the very compact form for the additional potential term

$$\Delta V(\bar{Q}) = \tfrac{1}{8}\hbar^2\, \Gamma^i_{jl}(\bar{Q})\, \Gamma^j_{im}(\bar{Q})\, g^{lm}(\bar{Q}) \ . \tag{3.23}$$

So our final path integral expression for the kernel in terms of the new variables Q_i reads

$$K(q'', t''; q', t') = [g(Q'')\, g(Q')]^{-1/4} \lim_{N\to\infty} \int \prod_{i=1}^{n} \prod_{k=1}^{N-1} \frac{\mathrm{d}Q_i(k)}{(2\pi i\hbar\epsilon)^{1/2}}$$
$$\times \prod_{k=0}^{N-1} g(\bar{Q}(k))^{1/2} \exp\left\{\frac{i}{\hbar} A_{\text{eff}}[\bar{Q}(k), \Delta Q(k)]\right\} \ , \tag{3.24}$$

with the following effective short time action:

$$A_{\text{eff}}[\bar{Q}(k), \Delta Q(k)] = \frac{1}{2\epsilon} \sum_{i,j} g_{ij}(\bar{Q}(k))\, \Delta Q_i(k)\, \Delta Q_j(k)$$
$$-\epsilon\, [V(F(\bar{Q}(k)) + \Delta V(\bar{Q}(k))] \ . \tag{3.25}$$

This is the main result of this section and the following observations are in order. First, the effective short time action (3.25) differs from the "naive" one by just the additional potenital term $\Delta V(Q)$ proportional to $\hbar^2$. This then explains why the formal change of variables breaks down starting at the two-loop level. Second, everywhere in the above path integral we have the midpoint coordinate $\bar{Q}_i(k)$ and not $Q_i(k+1)$ due to the fact that we adopted the symmetric definition for the path integral. Had we chosen to use.a different definition, the form of the additional potential term would naturally be different.

Next it is instructive to write down the equivalent phase space path integral which has the form

$$[g(Q'')g(Q')]^{-1/4} \lim_{N\to\infty} \int \prod_{i=1}^{n} \prod_{k=1}^{N-1} \mathrm{d}Q_i(k) \prod_{k=0}^{N-1} \frac{\mathrm{d}P_i(k)}{2\pi\hbar}$$

$$\times \exp\left\{\frac{i}{\hbar} \sum_{k=0}^{N-1} [P(k)\Delta Q(k) - H(P(k), \bar{Q}(k))]\right\} , \tag{3.26}$$

with the following classical Hamiltonian function

$$H(P, Q) = \tfrac{1}{2} \sum_{i,j=1}^{n} g^{ij}(Q) P_i P_j + V(F(Q)) + \Delta V(Q) . \tag{3.27}$$

We now observe that the operator Hamiltonian corresponding to the above phase space path integral contains non-commuting factors. That is precisely the reason why the careful treatment of this section was necessary. Namely, there is a unique ordering of non-commuting factors so that the final operator Hamiltonian describes the same quantum theory as the original one.

We can easily show that the operator Hamiltonian corresponding to the above midpoint phase space path integral is indeed the one with the correct ordering of factors. It can be found by performing a point transformation in operator formmalism with special care concerning the ordering of non-commuting operators. This discussion is presented in the appendix.

Finally we comment about the implications of the results of this section to the explicit Feynman diagram computations of sect. 2. We observe that the additional term $\Delta V(Q)$ does not depend on the potential and that it starts contributing in two-loop calculations. So, for example, it gives the following contribution to the two-loop vacuum energy:

$$-\tfrac{1}{8} i\hbar F^l_{,ij} F^i_{,jl} , \tag{3.28}$$

presicely canceling the sum of all two-loop bubble diagrams given by (2.19). This surely is an encouraging sign that the conclusions of this section are indeed correct.

4. Collective coordinates

Recently, the path integral method has been extensively used by us [1, 2] and several other authors [3–5] in order to formulate techniques for the study of collective phenomena in quantum field theory. The essential idea of these approaches consists in introducing new dynamical variables called the collective coordinates into the path integral expression for the transition amplitudes through δ-function gauge conditions. These δ-function conditions at the same time serve to eliminate the unwanted zero-frequency modes associated with the existence of continuous symmetries. Furthermore, in order to transfer the collective coordinates from the gauge conditions into the action a canonical transformation needs to be performed. This is a rather complicated non-linear transformation and in previous discussions it was performed formally, namely as the one in sect. 2 of this paper. It is this last step which requires more care and now we would like to discuss this problem giving the improved treatment.

We have learned in sect. 3 that careful treatment of point transformations in path integral leads to additional potential terms in the action as compared to the formal change of integration variable. Then, due to the same reason, additional terms will appear in the path integral collective coordinate method also ★.

To be specific, we consider the example of the one-soliton sector case where the collective coordinate is the center-of-mass variable introduced in order to maintain the translational invariance of the theory. For a detailed discussion and the development of a systematic perturbation expansion about the one-soliton state, the reader is referred to refs. [1, 3]. Here we will not repeat these derivations but just discuss the modifications to the more careful treatment. Namely, we will present the calculation of the additional potential term for the one-soliton case which is a direct application of the results derived in sect. 3.

We introduce the collective coordinate $X(t)$ through the following change of variables:

$$\phi(x, t) = \phi_0(x - X(t)) + \chi(x - X(t), t) \;, \tag{4.1}$$

where $\phi_0(x - X)$ is the classical soliton solution. The χ field describing the small oscillations satisfies the gauge condition

$$\int \mathrm{d}x \; \psi_0(x - X(t)) \; \chi(x - X(t), t) = 0 \;, \tag{4.2}$$

where $\psi_0(x)$ stands for the zero-frequency wave function

$$\psi_0(x) = \left(\int \mathrm{d}x \; \phi_0'(x)^2\right)^{-1/2} \phi_0'(x) \;, \tag{4.3}$$

★ The fact that higher order infinitesimal terms can contribute when introducing collective coordinates in path integral was pointed out to us by Professor R. Jackiw. Recently this was also emphasized in the work of R. Rajaraman and E. Weinberg [13].

and it is the lowest energy wave function in a complete set $\{\psi_i(x); i = 0, 1, 2, ..., \}$. We then write the following expression for the χ field

$$\chi(x - X(t), t) = \sum_{n=1}^{\infty} \psi_n(x - X(t))\, Q_n(t) \ , \tag{4.4}$$

leaving out the zero-frequency mode due to the gauge condition (4.2).

We denote $X(t) = Q_0(t)$ and then using the notation of sect. 3 we have

$$F^x(Q) = \phi_0(x - Q_0(t)) + \sum_{n=1}^{\infty} \psi_n(x - Q_0(t)) Q_n(t) \ , \tag{4.5}$$

and so

$$g_{ij}(Q) = \int \mathrm{d}x\, F^x_{,i}(Q)\, F^x_{,j}(Q) \ ,$$

$$\sum F^a_{,i}(Q) F^a_{,jlm}(Q) = \int \mathrm{d}x F^x_{,i}(Q)\, F^x_{,jlm}(Q) \, , \qquad i, j, l, m = 0, 1, 2, ... \, . \tag{4.6}$$

The matrix $g_{ij}(Q)$ is explicitly given by

$$g_{00}(Q) = \int \mathrm{d}x\, (\phi_0'(x) + \chi'(x, t))^2 = (\phi', \phi') \ ,$$

$$g_{0n}(Q) = g_{n0}(Q) = -\int \psi_n(x)(\phi_0'(x) + \chi'(x, t)) = -(\psi_n, \phi') \ ,$$

$$g_{nn'}(Q) = \delta_{nn'} \qquad n, n' = 1, 2, ... \, , \tag{4.7}$$

and its determinant is easily calculated to be

$$g(Q) = \det g_{ij}(Q) = (\psi_0, \phi')^2 \ . \tag{4.8}$$

We also need the inverse matrix $g^{ij}(Q)$ which can be read off directly from the phase-space path integral derived in our earlier papers

$$g^{00}(Q) = \frac{1}{(\psi_0, \phi')^2} \ ,$$

$$g^{0n}(Q) = g^{n0}(Q) = \frac{(\psi_n \phi')}{(\psi_0, \phi')^2} \ ,$$

$$g^{nn'}(Q) = \delta_{nn'} + \frac{(\psi_n, \phi')(\phi', \psi_{n'})}{(\psi_0, \phi')^2} \ . \tag{4.9}$$

Next we simply use the general formula (3.19) which was obtained in sect. 3 to calculate the additional potential term for this specific case. After some calculation we find *

$$\Delta V(\chi) = \tfrac{1}{8}\hbar\left[-\frac{3(\psi_0', \psi_0')}{(\psi_0, \phi')^2} + 2\frac{(\psi_0', \phi'')}{(\psi_0, \phi')^3} + \frac{(\psi_0', \phi')^2}{(\psi_0, \phi')^4} - \frac{\sum |\psi_n, \psi_m'|^2}{(\psi_0, \phi')}\right]. \tag{4.10}$$

This additional potential term differs from the one found by Tomboulis [6] who used the operator collective coordinate method, but that is not contradictory as will be explained later.

Now, the one-soliton generating functional is expressed in the following phase-space path integral form

$$Z(J; p', p) = \int DPQX\, e^{-ip'X(+\infty)}\, e^{ipX(-\infty)} \int D\pi D\chi\, \delta\left(\int dx\, \psi_0 \chi\right)$$

$$\times\, \delta\left(\int dx\, \psi_0 \pi\right)\left\{\exp \frac{i}{\hbar}\int dt\left[P(t)\dot{X}(t) + \int dx\, \pi(x,t)\,\dot{\chi}(x,t)\right.\right.$$

$$\left.\left. -H_{\text{eff}}(P, \pi, \chi) + \int dx J(x + X(t), t)(\chi(x,t) + \phi_0(x)\right]\right\}, \tag{4.11}$$

with the understanding that this is a symmetric path integral, namely the one with the additional specification, that the short time action is to be evaluated at the midpoint. The total effective Hamiltonian reads

$$H_{\text{eff}}(P, \pi, \chi) = \frac{(P + \int \pi\chi'\, dx)^2}{2(\psi_0, \phi')^2} + \sum_{l=2} \frac{1}{l!} V^{(l)}(\phi_0)\chi^l + \Delta V(\chi)\ . \tag{4.12}$$

In comparison with the effective Hamiltonian derived before by making formal changes of variables in path integral it contains the additional potential term $\Delta V(\chi)$. Since it is proportional to $\hbar^2$ is starts contributing at the two-loop level and thus all our previous calculations which were performed at tree and one-loop approximation remain valid.

An example where this new term becomes relevant is the calculation of the two-loop quantum correction to the soliton mass. For the $\lambda\phi^4$ besides all the two-loop bubble diagrams which contribute, there is also the additonal terms

$$\frac{\hbar^2}{8M_0^2}\int dx\, \phi_0''^2(x) \tag{4.13}$$

coming from $\Delta V(\chi)$. After some calculation we obtain the following expression

* We are grateful to De Vega for pointing out an error in our initial calculation.

of the total two-loop mass correction

$$M^{(2)} = \tfrac{1}{8}\lambda \int dx G(x,x)G(x,x) - \tfrac{1}{2}\lambda^2 \int dx dy \phi_0(x) G(x,x) G^{(2)}(x,y) G(y,y)\phi_0(y)$$

$$-\tfrac{1}{48}\lambda^2 \sum_{l,m,n} \frac{(\int dx \phi_0 \psi_l \psi_m \psi_n)^2}{\omega_l + \omega_m + \omega_n} + \frac{1}{8M_0}\int dx dy \left(\frac{\partial}{\partial x}\frac{\partial}{\partial y} G(x,y)\right) G^{-1}(x,y)$$

$$-\frac{1}{8M_0^2}\int dx \phi_0''^2(x) \quad -\frac{1}{8}\frac{\sum_{m,n} |\psi_n, \psi_m'|^2}{(\psi_0, \phi_0')^2} \,. \tag{4.14}$$

Here we used the Feynman rules and their graphical representation as described in our earlier paper [1]. We denoted

$$G(x,y) = \sum_{n=1}^{\infty} \psi_n(x) \frac{1}{2\omega_n} \psi_n^*(y), \qquad G^{(l)}(x,y) = \sum_{n=1} \psi_n(x)(2\omega_n)^{-l} \psi_n^*(y) \,. \tag{4.15}$$

The final expression (4.14) we found is exactly the same answer * as the one obtained in ref. [14] using the method of Goldstone and Jackiw [15] and also the operator collective coordinate method [6].

It is not difficult to see that our path integral for the one-soliton sector corresponds to the operator theory with correctly ordered non-commuting factors in the Hamiltonian. Namely, due to the fact that in (4.11) we have the symmetric ("midpoint") path integral we conclude that the corresponding operator Hamiltonian reads

$$\hat{H} = \tfrac{1}{2}\left\{\left(\hat{P} + \int dx \hat{\pi}\hat{\chi}\right)^2, \frac{1}{(\psi_0, \phi')^2}\right\}_W + \sum_{l=2} \frac{1}{l!} V^{(l)}(\phi_0)\hat{\chi}^l + \Delta V(\hat{\chi}) \;, \tag{4.16}$$

the non-commuting factors being Weyl ordered. By reordering the first term one can easily show that this Hamiltonian operator coincides with the one derived in the operator collective coordinate approach.

In general, a similar discussion as the one given in this section for the one-soliton sector applies also to the general case when there are more collective coordinates like for example in field theories with internal symmetry [13]. Using the general formula derived in sect. 3 one can always calculate the additional potential term. Then one can directly read off the correct Feynman rules from the effective action. At the end of this section we like to emphasize once again that it is now possible to treat in path integral such delicate problems as ordering of factors and also derived the correct Feynman rules for theories when the corresponding operator Hamiltonian containes non-commuting factors. Consequently, this then establishes the path integral collective method at the same level of rigor as the operator approach.

* Actually, ref. [14] in its initial version, does not agree with (4.14). After our result appeared L. Jacobs (private communication) found an error in his computation such that his results now agree with ours.

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Appendix

In this appendix we demosntrate that the results we have found in sect. 3 are in precise agreement with the parallel derivations given in the operator approach. First in order to find the operator Hamiltonian corresponding to the path integral (3.26) we must know the correspondence between operator Hamiltonians and path integrals. This question was investigated carefully by Berezin and we use his results here. For more details the reader is referred to his work [10] or ref. [16].

The result of Berezin which is of immediate relevance to us can be summarized by the statement that the operator Hamiltonian describing the same quantum theory as the symmetric (midpoint) path integral is the one where the non-commuting factors are Weyl ordered. Weyl ordering is defined by the following relation.

$$(\alpha\hat{p} + \beta\hat{q})^N = \sum_{k,l} \alpha^k \beta^l \{\hat{p}^k, \hat{q}^l\}_W \frac{N!}{l!k!} . \tag{A.1}$$

The above correspondence follows from the identity

$$\langle q'' | \{\hat{p}^k, \hat{q}^l\}_W | q' \rangle = \int \frac{\mathrm{d}p}{2\pi\hbar} e^{ip(q''-q')/\hbar} [p^k(\tfrac{1}{2}(q''+q'))^l] , \tag{A.2}$$

which can easily be proved.

So the operator Hamiltonian corresponding to our symmetric path integral (3.26) reads

$$\hat{H} = \tfrac{1}{2} \{\hat{P}_i \hat{P}_j g^{ij}(\hat{Q})\}_W + V(F(\hat{Q})) + \Delta V(\hat{Q}) . \tag{A.3}$$

Next we perform a point transformation in the operator formalism. Starting with the original Hamiltonian in coordinate representation

$$-\tfrac{1}{2}\hbar^2 \sum_{a=1}^{n} \frac{\partial^2}{\partial q_a^2} + V(q) , \tag{A.4}$$

and making the change of variables $q_a = F^a(Q)$ we get for this differential operator

$$-\tfrac{1}{2}\hbar^2 \sum_{i,j=1}^{n} \frac{1}{\sqrt{g(Q)}} \frac{\partial}{\partial Q_i} g^{ij}(Q) g(Q)^{1/2} \frac{\partial}{\partial Q_j} + V(F(Q)) \ , \tag{A.5}$$

with the following scalar product

$$(\psi_1, \psi_2) = \int \prod_a \mathrm{d}Q_a g(Q)^{1/2} \psi_1^*(F(Q))\, \psi_2(F(Q)) \ . \tag{A.6}$$

Redefining the Hilbert space so as to eliminate the measure from this scalar product we write the operator Hamiltonian (3.30) in the form

$$\hat{H} = \tfrac{1}{2} g(\hat{Q})^{-1/4} \sum_{i,j=1}^{n} \hat{P}_i g^{ij}(\hat{Q}) g(\hat{Q})^{1/2} \hat{P}_j g(\hat{Q})^{-1/4} + V(F(\hat{Q})) \ . \tag{A.7}$$

Finally we order the non-commuting factors in (3.28) and using the relation

$$\hat{P}_i g^{ij}(\hat{Q}) \hat{P}_j = \{\hat{P}_i g^{ij}(\hat{Q}) \hat{P}_j\}_{\mathrm{W}} + \tfrac{1}{4}\hbar g^{ij}(\hat{Q})_{,ij} \ , \tag{A.8}$$

rewrite the operator Hamiltonian (3.32) as

$$\hat{H} = \tfrac{1}{2} \{\hat{P}_i \hat{P}_j g^{ij}(\hat{Q})\}_{\mathrm{W}} + V(F(\hat{Q})) + \Delta V'(\hat{Q}) \ , \tag{A.9}$$

with

$$\Delta V'(\hat{Q}) = \tfrac{1}{4}\hbar[\tfrac{1}{2} g^{ij}(\hat{Q})_{,ij} - 2g(\hat{Q})^{-1/4}(g^{ij}(\hat{Q}) g^{1/2}(g(\hat{Q})^{-1/4})_{,i})_{,j}] \ . \tag{A.10}$$

Now it is straightforward to show that the additional potential terms $\Delta V(Q)$ and $\Delta V'(Q)$ are indeed equal which finishes the proof that our final path integral corresponds to the correctly ordered operator theory.

References

[1] J.-L. Gervais and B. Sakita, Phys. Rev. D11 (1975) 2950;
J.-L. Gervais, A. Jevicki and B. Sakita, Phys. Rev. D12 (1975) 1038.
[2] J.-L. Gervais, A. Jevicki and B. Sakita, Phys. Reports 23 (1976) 281.
[3] C. Callan and D. Gross, Nucl. Phys. B93 (1975) 29.
[4] L.D. Faddeev, IAS preprint (1975).
[5] A. Hosoya and K. Kikkawa, Nucl. Phys. B101 (1975) 271.
[6] E. Tomboulis, Phys. Rev. D12 (1975) 1678.
[7] N. Christ and T.D. Lee, Phys. Rev. D12 (1975) 1606;
M. Creutz, Phys. Rev. D12 (1975) 3126.

[8] L.D. Faddeev, Theor. Math. Phys. 1 (1969) 3;
S. Coleman, Secret symmetry, 1973 Erice lectures.
[9] G,'t Hooft and M. Veltman, Diagrammar, CERN report 73-9 (1973).
[10] F.A. Berezin, Theor, Math. Phys. 6 (1971) 194.
[11] S.F. Edwards and Y.V. Gulyaev, Proc. Roy. Soc. A279 (1964) 229.
[12] D. McLaughlin and L.S. Schulman, J. Math. Phys. 12 (1971) 2520.
[13] R. Rajaraman and E. Weinberg, Phys. Rev. D11 (1975) 2950.
[14] L. Jacobs, MIT preprint, Center for Theoretical Physics publication no. 499, 1975.
[15] J. Goldstone and R. Jackiw, Phys. Rev. D11 (1975) 1486.
[16] M. Mizrahi, J. Math. Phys. 16 (1975) 2201.