QFT 1002 renormalization and spontaneous symmetry breaking

Pok Man Lo

University of Pittsburgh

QFT 1002, 1-29-2009

personal information

- institution: University of Pittsburgh
- advisor: Eric Swanson

outline

1. review of renormalization
2. spontaneous symmetry breaking

outline

1. *review of renormalization*
2. *spontaneous symmetry breaking*

definition of N point function

For a free scalar field, we know that the propagator is given by

$$< T\{\phi_x \phi_y\} >= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2 - m^2 + i\epsilon}$$

definition of N point function

For a free scalar field, we know that the propagator is given by

$$< T\{\phi_x \phi_y\} >= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2 - m^2 + i\epsilon}$$

this is an example of 2-point function.

In general, we define $< T\{\phi_1 \phi_2 ...\} >$ as N-point function.
note that an N-point function, with N space-time points, can only have $N - 1$ degree of freedom due to translational invariance.

definition of N point function

For a free scalar field, we know that the propagator is given by

$$< T\{\phi_x \phi_y\} >= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2 - m^2 + i\epsilon}$$

this is an example of 2-point function.

In general, we define $< T\{\phi_1 \phi_2 ...\} >$ as N-point function.
note that an N-point function, with N space-time points, can only have $N - 1$ degree of freedom due to translational invariance.
$< T\{\phi_x \phi_y\} >$ only depends on $(x - y)$ and hence its fourier transform depends only on one k

definition of N point function

likewise, the fourier transform of $< T\{\phi_1\phi_2\phi_3\phi_4\} >$ can only depends on 3 k's , e.g the s, t and u channel

The other way to motivate the definition of N-point functions is from the language of generating functional.
Recall the generating functional is defined as

$$\mathcal{Z} = \int D\phi e^{i\int(\mathcal{L}+j\phi)} = e^{i\mathcal{W}}$$

it is clear that

$$< T\{\phi_1\phi_2 ...\} >= \frac{1}{\mathcal{Z}}(-i\frac{\delta}{\delta j_1}...)\mathcal{Z}$$

definition of N point function

it is non-trivial, but true, that the connected N-point function is given by

$$< T\{\phi_1\phi_2...\} >_c = i(-i\frac{\delta}{\delta j_1}...)\mathcal{W}$$

For the free case, it's easy to check all that:

$$\begin{aligned}\mathcal{Z} = & \int D\phi e^{i\int(-\phi\frac{1}{2}(\partial^2+m^2)\phi+j\phi)} \\ & = \mathcal{Z}_0 e^{i\frac{1}{2}\int j\frac{1}{\partial^2+m^2}j}\end{aligned}$$

definition of N point function

$$\mathcal{W} = \frac{1}{2}\int j \frac{-1}{k^2 - m^2} j$$

and

$$\frac{\delta^2 \mathcal{W}}{\delta j \delta j} <-> \frac{1}{\partial^2 + m^2} <-> i < T\{\phi_x \phi_y\} >$$

definition of N point function

$$\mathcal{W} = \frac{1}{2}\int j \frac{-1}{k^2 - m^2} j$$

and

$$\frac{\delta^2 \mathcal{W}}{\delta j \delta j} <-> \frac{1}{\partial^2 + m^2} <-> i < T\{\phi_x \phi_y\} >$$

In general, the N point functions would be very messy. Being able to find all the N point functions is equivalent to being able to solve the whole quantum field theory. Of course it's only possible for some simple cases.
However, we can derive exact relations among N point functions, these techniques go under the name of Schwinger Dyson Equation, which will not be discussed here.

review of renormalization

The renormalization program is the observation that divergent integrals in a QFT can be systematically absorbed, such that $< T\{\phi_{phy1}\phi_{phy2}...\} >$ makes **physical sense**

review of renormalization

The renormalization program is the observation that divergent integrals in a QFT can be systematically absorbed, such that $< T\{\phi_{phy1}\phi_{phy2}...\} >$ makes **physical sense**

for example, we insist, for $p^2 \approx m_{phy}^2$,

$$< T\{\phi_{phy1}\phi_{phy2}\} >= \frac{i}{p^2 - m_{phy}^2} + ...$$

this is called a **renormalization condition**

review of renormalization

To make contact with our theory, we write

$$\phi_0 = \sqrt{Z_\phi}\phi_{phy}$$

relating ϕ_0 defined from our theory to the physical 2 point functions
such that

$$< T\{\phi_{phy1}\phi_{phy2}\} >= \frac{1}{Z_\phi} < T\{\phi_0\phi_0\} >$$

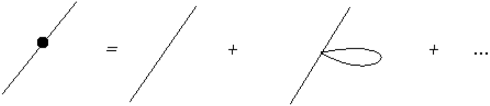
note that $< T\{\phi_0\phi_0\} >$ can be computed from our theory.

review of renormalization

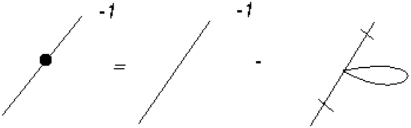
Consider a ϕ^4 theory:

$$\mathcal{L} = \frac{1}{2}(\partial\phi_0)^2 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4$$

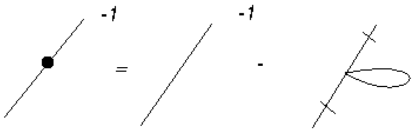
to one loop:



or



review of renormalization



rhs:

$$\frac{p^2 - m_0^2 - \frac{i}{2}\lambda_0 \int \frac{d^4 l}{(2\pi)^4} \frac{i}{l^2 - m_0^2 + i\epsilon}}{i}$$

while

lhs:

$$[< T\{\phi_0 \phi_0\} >]^{-1} = \frac{1}{Z} \frac{1}{< T\{\phi_{phy} \phi_{phy}\} >}$$

review of renormalization

As have been explained:

$$< T\{\phi_{phy}\phi_{phy}\} >= \frac{i}{p^2 - m_{phy}^2 - \Sigma_{phy}^2(p^2)}$$

with

$$\begin{aligned} \Sigma_{phy}^2(p^2)|_{p^2\to m_{phy}^2} &= 0. \\ \frac{d}{dp^2}\Sigma_{phy}^2(p^2)|_{p^2\to m_{phy}^2} &= 0. \end{aligned}$$

That is, lhs is completely specified by the renormalization conditions.

finally,

$$\frac{p^2 - m_{phy}^2 - \Sigma_{phy}^2(p^2)}{Z_\phi} = p^2 - m_0^2 - \frac{i}{2}\lambda_0 \int \frac{d^4l}{(2\pi)^4} \frac{i}{l^2 - m_0^2 + i\epsilon}$$

using the renormalization conditions, we find:

$$\begin{aligned} Z_\phi &= 1 \\ \Sigma_{phy}^2(p^2) &= 0. \\ &\forall p^2 \end{aligned}$$

and ...

review of renormalization

$$m_{phy}^2 = m_0^2 + \frac{i}{2}\lambda_0 \int \frac{d^4 l}{(2\pi)^4} \frac{i}{l^2 - m_0^2 + i\epsilon}$$

review of renormalization

$$m_{phy}^2 = m_0^2 + \frac{i}{2}\lambda_0 \int \frac{d^4 l}{(2\pi)^4} \frac{i}{l^2 - m_0^2 + i\epsilon}$$

two comments are in order:

- in general (as in QED), $Z \neq 1$, $\Sigma \neq 0$
- m_0^2 has to be divergent for the above to be true

studying the 4 point function, we find that λ_0 has to be divergent too.

the birth of counter terms

observe that we have:

$$
\begin{aligned}
m_{phy}^2 &= m_0^2 Z - \delta m^2 \\
\lambda_{phy} &= \lambda_0 Z^2 - \delta\lambda \\
1 &= Z - \delta Z
\end{aligned}
$$

lhs $->$ specified by renormalization conditions rhs $->$ specified by bare theory
This motivates yet another way of approaching renormalization in QFT: the counter term method

the birth of counter terms

Starting from the bare theory:

$$\mathcal{L} = \frac{1}{2}(\partial\phi_0)^2 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4$$

we can rewrite $m_0, \phi_0, \lambda_0, Z$ in terms of $m_{phy}, \phi_{phy}, \lambda_{phy}, 1$

$$\begin{aligned}\mathcal{L} \rightarrow \quad & \frac{1}{2}(\partial\phi_{phy})^2 - \frac{1}{2}m_{phy}^2\phi_{phy}^2 - \frac{\lambda_{phy}}{4!}\phi_{phy}^4 + \\ & (\delta Z)\frac{1}{2}(\partial\phi_{phy})^2 - \frac{1}{2}\delta m^2\phi_{phy}^2 - \delta\lambda\frac{1}{4!}\phi_{phy}^4\end{aligned}$$

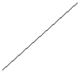
note that the two Lagrangian are **equivalent**

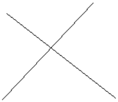
the birth of counter terms

we can read off the Feynman rules:
the old ones:

$$\frac{i}{p^2 - m_{phy}^2 + i\epsilon}$$

$$-i\lambda_{phy}$$





the birth of counter terms

$$\mathcal{L} \rightarrow \quad \frac{1}{2}(\partial\phi_{phy})^2 - \frac{1}{2}m_{phy}^2\phi_{phy}^2 - \frac{\lambda_{phy}}{4!}\phi_{phy}^4 + $$
$$(\delta Z)\frac{1}{2}(\partial\phi_{phy})^2 - \frac{1}{2}\delta m^2\phi_{phy}^2 - \delta\lambda\frac{1}{4!}\phi_{phy}^4$$

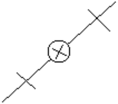
the birth of counter terms

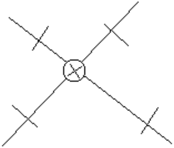
$$\mathcal{L} \rightarrow \quad \frac{1}{2}(\partial\phi_{phy})^2 - \frac{1}{2}m_{phy}^2\phi_{phy}^2 - \frac{\lambda_{phy}}{4!}\phi_{phy}^4 + (\delta Z)\frac{1}{2}(\partial\phi_{phy})^2 - \frac{1}{2}\delta m^2\phi_{phy}^2 - \delta\lambda\frac{1}{4!}\phi_{phy}^4$$

the new ones:

$$i[\delta Z p^2 - \delta m^2]$$

$$-i\delta\lambda$$





the birth of counter terms

in this picture, if we are to compute the two point function, it is by definition:

$$< T\{\phi_{phy}\phi_{phy}\} >$$

and we would have...

lhs $->$ easy application of renormalization condition:

$$< T\{\phi_{phy}\phi_{phy}\} >= \frac{i}{p^2 - m_{phy}^2 - \Sigma_{phy}^2(p^2)}$$

the birth of counter terms

of course, the same "miracle" happens: δm^2 is chosen to absorb the infinity!

outline

1. review of renormalization
2. spontaneous symmetry breaking

symmetry in field theory

a classical symmetry is such that the Lagrangian is invariant under the symmetry operation: $\mathcal{L} \rightarrow \mathcal{L}$
for example:

- Lorentz symmetry $e^{-iJ\omega}$
- Gauge symmetry $e^{-ig\alpha}$
- chiral symmetry $e^{-i\alpha\gamma_5}$

symmetry in field theory

A symmetry is characterized by a generator
in Gauge symmetry: $e^{-ig\alpha}$
where

$$\alpha = \vec{\alpha} \cdot \vec{T}$$

here $\vec{T}$'s are generators, satisfying some commutation relation.
For a SU(2) symmetry, $\vec{T} <-> \frac{1}{2}\vec{\sigma}$ satisfying

$$[T_i, T_j] = i\epsilon_{ijk}T_k$$

Note that the generators are 'matrix' acting on 'vector', e.g
$\vec{\tau}$ acting on isospin doublet $[p, n]$

symmetry in field theory

There is a corresponding conserving Noether current:

$$J^\mu <-> [\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_i} \delta \phi_i]$$

e.g
chiral symmetry: $e^{-i\alpha\gamma_5}$

$$J^\mu <-> \overline{\psi}\gamma^\mu\gamma_5\psi$$

such that

$$\partial_\mu J^\mu = 0.$$

symmetry in field theory

To break a symmetry, we can:

- explicit breaking: add a term that is not invariant under the symmetry operation
  e.g $-m\bar{\psi}\psi$ term in $\mathcal{L}$ in chiral symmetry
- spontaneous breaking: the symmetry is NOT broken at all! It is hidden. The symmetry can still be realized by the conserving Noether current

in the case of spontaneous symmetry breaking (also called dynamical symmetry breaking), it is the **vacuum** that does not respect the symmetry.

spontaneous symmetry breaking

we consider a simple example

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

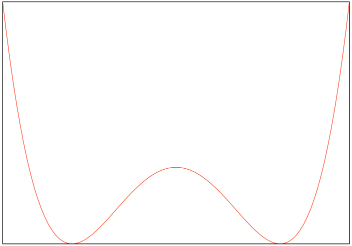
spontaneous symmetry breaking

we consider a simple example

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

A wrong sign in the mass term.
The Lagrangian can be written as $T - V$, with

$$V[\phi] = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4$$



spontaneous symmetry breaking

The minimum is at $\pm\sqrt{\frac{6\mu^2}{\lambda}}$
we can expand ϕ around this minimum:

$$\phi = v + \phi'$$

hence we can use ϕ' field to study the system:
The Lagrangian becomes:

$$\begin{aligned}\mathcal{L}[\phi'] = \quad & \frac{1}{2}(\partial\phi')^2 + [\frac{1}{2}\mu^2 v^2 + \frac{\lambda}{4!}v^4] + \\ & 0\phi' + \\ & [\frac{1}{2}\mu^2 - \frac{\lambda}{4!}6v^2]\phi'^2 \\ & -\frac{\lambda}{4!}4v\phi'^3 - \frac{\lambda}{4!}\phi'^4\end{aligned}$$

spontaneous symmetry breaking

finally:

$$\mathcal{L}[\phi'] = \frac{1}{2}(\partial\phi')^2 - \frac{1}{2}(2\mu^2)\phi'^2 - \sqrt{\frac{\lambda}{6}}\mu\phi'^3 - \frac{\lambda}{4!}\phi'^4$$

here we have

- the right sign for the mass term: $2\mu^2$
- new ϕ'^3 interation
- old ϕ'^4 interation

It is clear that $\phi' -> -\phi'$ symmetry is no longer explicitly present. It is "hidden" in the **relation** between ϕ'^3 and ϕ'^4 interations.

spontaneous symmetry breaking

we consider another related example

$$\mathcal{L} = \frac{1}{2}(\partial\vec{\phi})^2 + \frac{1}{2}\mu^2\vec{\phi}^2 - \frac{\lambda}{4}[\vec{\phi}^2]^2$$

note:

- there is a $O(N)$ symmetry, it's a continuous symmetry
- we can do the same trick, shifting the vector this time: $\vec{\phi_0}^2 = \frac{\mu^2}{\lambda}$
- the breaking condition only specify the magnitue, not the direction.

The last point is essential for the existence of Goldstone boson: a massless mode

spontaneous symmetry breaking

spontaneous symmetry breaking

now I will illustrate the $N = 2$ case on board

spontaneous symmetry breaking

now I will illustrate the $N = 2$ case on board
by selecting the $\vec{\phi}_0$ along the N-th direction, the remaining $N - 1$ degree of freedom all becomes Goldstone bosons
observe that the vacuum vector do not respect the rotation along those $N - 1$ axes: it seems that each generator it breaks will give you one goldstone boson
it turns out it is generally true.

Goldstone theorem

proof of Goldstone theorem:

consider a potential $V[\vec{\phi}]$:
let $\vec{\phi_0}$ be the vac state such that:

$$\frac{\partial}{\partial \phi_a} V|_{\vec{\phi}\to\vec{\phi_0}} = 0$$

expanding V around this vac, we fine

$$V[\vec{\phi}] = V[\vec{\phi_0}] + 0 + \frac{1}{2}(\vec{\phi} - \vec{\phi_0})_a(\vec{\phi} - \vec{\phi_0})_b \frac{\partial^2}{\partial \phi_a \partial \phi_b} V|_{\vec{\phi}\to\vec{\phi_0}} + ...$$

note that $\frac{\partial^2}{\partial \phi_a \partial \phi_b} V|_{\vec{\phi}\to\vec{\phi_0}}$ corresponds to mass terms

Goldstone theorem

if there is a symmetry operation: $\vec{\phi} \to \vec{\phi} + \alpha\vec{\Delta}[\vec{\phi}]$
such that
$V[\vec{\phi}] = V[\vec{\phi} + \alpha\vec{\Delta}[\vec{\phi}]]$

it means:
$\Delta^a[\vec{\phi}]\frac{\partial}{\partial\phi^a}V = 0.$

further differentiating gives:

$$\Delta^a[\vec{\phi_0}]\frac{\partial^2}{\partial\phi^a\partial\phi_b}V|_{\vec{\phi}\to\vec{\phi_0}} = 0.$$

it shows that $\frac{\partial^2}{\partial\phi^a\partial\phi_b}V|_{\vec{\phi}\to\vec{\phi_0}}$ is zero as long as $\Delta^a[\vec{\phi_0}] \neq 0$, that is, if the vac **does not** respect the symmetry

Goldstone theorem

- $N = 3$ case:
  The vac is picked to be along 3-direction, so it's not invariant under rotation along 1 and 2-axis.
  There are two goldstone bosons.
- $N = 2$ case:
  The vac is picked to be along 2-direction, there is only one generator for this group.
  There are one goldstone boson.

spontaneous symmetry breaking and renormalization

Before the spontaneous symmetry breaking, we have:

$$\mathcal{L} = \frac{1}{2}(\partial\vec{\phi})^2 + \frac{1}{2}\mu^2\vec{\phi}^2 - \frac{\lambda}{4}[\vec{\phi}^2]^2$$

after breaking, we have:

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}(\partial_\mu \pi^k)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 \\ &- \sqrt{\lambda}\mu\sigma^3 - \sqrt{\lambda}\mu(\pi^k)^2\sigma - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}(\pi^k)^2\sigma^2 - \frac{\lambda}{4}\left[(\pi^k)^2\right]^2.\end{aligned}$$

spontaneous symmetry breaking and renormalization

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \pi^k)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2$$
$$- \sqrt{\lambda}\mu\sigma^3 - \sqrt{\lambda}\mu(\pi^k)^2\sigma - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}(\pi^k)^2\sigma^2 - \frac{\lambda}{4}\left[(\pi^k)^2\right]^2.$$

here I illustrate a few Feynman's rule:

- sigma propagator: $\frac{i}{p^2-(2\mu^2)}$
- 'pion' propagator: $\frac{i}{p^2}\delta^{ij}$
- sigma three point interation: $-6i\mu\sqrt{\lambda}$
- ...

spontaneous symmetry breaking and renormalization

again, from

$$\mathcal{L} = \frac{1}{2}(\partial\vec{\phi})^2 + \frac{1}{2}\mu^2\vec{\phi}^2 - \frac{\lambda}{4}[\vec{\phi}^2]^2$$

we can only have three counter terms:
$\delta Z(\partial\vec{\phi})^2$, $\delta\mu^2(\vec{\phi})^2$ and $\delta\lambda[(\vec{\phi})^2]^2$
again, we need to rewrite it in the shifted field representation:
the result:

$$\frac{\delta_Z}{2}(\partial_\mu\pi^k)^2 - \frac{1}{2}(\delta_\mu + \delta_\lambda v^2)(\pi^k)^2 + \frac{\delta_Z}{2}(\partial_\mu\sigma)^2 - \frac{1}{2}(\delta_\mu + 3\delta_\lambda v^2)\sigma^2$$
$$- (\delta_\mu v + \delta_\lambda v^3)\sigma - \delta_\lambda v\sigma(\pi^k)^2 - \delta_\lambda v\sigma^3$$
$$- \frac{\delta_\lambda}{4}\left[(\pi^k)^2\right]^2 - \frac{\delta_\lambda}{2}\sigma^2(\pi^k)^2 - \frac{\delta_\lambda}{4}\sigma^4.$$

finally, we need the renormalization condition: we expect three of them!
we can pick: 2 from the sigma 2 point function, plus 1 from sigma 4 point function
or
1 from one point function of sigma (a new feature from the theory!), 1 from sigma 2 point function and 1 from sigma 4 point function

summary

- we review the definition of N point functions and renormalization theory
- we study spontaneous symmetry breaking and the Goldstone theorem for massless mode
- we study the Feynman rules of the symmetry broken theory and study the renormalization aspect of it

more to go...

- chiral symmetry, the story of pion
- quantum fluctuation breaks the symmetry spontaneously
- explicit symmetry breaking and the Noether theorem
- effective potential approach, generating functional techniques
- the physics of renormalization: effective field theory