QFT2 — assignment 1: Dirac Basics

1. **Diracology**

Using $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ prove
(i) $\{\gamma_5, \gamma_\mu\} = 0$
(ii) $\not p \not p = p^2$
(iii) $Tr(\not a \not b) = 4a \cdot b$.
(iv) $Tr(\text{odd no } \gamma' s) = 0$. (Hint, insert $\gamma_5^2 = 1$ in the trace.)
(v) $Tr(\not a \not b \not c \not d) = 4[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)]$ (Hint: use $\not a \not b + \not b \not a = 2a \cdot b$ repeatedly.)
(vi) $Tr(\gamma_5) = 0$.
(vii) $Tr(\gamma_5 \not a \not b) = 0$
(viii) $Tr(\gamma_5 \not a \not b \not c \not d) = 4i\epsilon_{\mu\nu\lambda\sigma} a^\mu b^\nu c^\lambda d^\sigma$.
(ix) $\gamma_\mu \gamma^\mu = 4$
(x) $\gamma_\mu \not a \gamma^\mu = -2 \not a$
(xi) $\gamma_\mu \not a \not b \gamma^\mu = 4a \cdot b$
(xii) $\gamma_\mu \not a \not b \not c \gamma^\mu = -2 \not c \not b \not a$

2. **conservation of total spin**
Show that $[H, \vec{J}] = 0$ where $H = \alpha \cdot \vec{p} + \beta m$ and $\vec{J} = \vec{L} + \frac{1}{2}\vec{\Sigma}$ with $\vec{\Sigma} = diag(\vec{\sigma}, \vec{\sigma})$.

3. **Lorentz invariance**
(i) Show $S^{-1}\gamma^\mu S = \Lambda^\mu_{\cdot\nu}\gamma^\nu$ to first order. Thus γ^μ transforms as a 4-vector.
(ii) Prove $S^{-1} = \gamma^0 S^\dagger \gamma^0$.
(iii) Prove $\gamma^5 S = S\gamma^5$.

4. **Parity**

(i) Give the explicit transformation properties of all bilinears under parity.

(ii) People often wonder how the reaction $\pi \to \mu\nu$ can proceed since the pion has $J = 0$ while the W is a vector ($J = 1$) particle. Explain. Note that the current associated with the W is proportional to $V^\mu - A^\mu = \bar{\psi}\gamma^\mu(1 - \gamma_5)\psi$.

5. **Bilinears**

(i) Prove that the operator $\bar{\psi}\psi$ is a scalar by establishing the quantum numbers of the state $\int d^3x \bar{\psi}(\vec{x})\psi(\vec{x}) |0\rangle$ in the nonrelativistic limit. Work in the rest frame of the created particle.

(ii) Do the same for the vector current. What happens for $\int d^3x \bar{\psi}\gamma^0\psi |0\rangle$?

Hint: a fermion-antifermion in the state ${}^{2S+1}L_J$ has parity $P = (-1)^{L+1}$ and charge conjugation $C = (-1)^{L+S}$. Prove these relations for bonus points.

6. **Chirality and Helicity**

(i) Show that chirality is not a good quantum number for a massive fermion.

(ii) Show that helicity is conserved for a massive fermion.

(iii) Show that $\gamma_5 u = \Sigma \cdot \hat{p} u$ in the high energy limit where $\Sigma = diag(\sigma, \sigma)$. Thus chirality and helicity are the same in the high energy limit.