1. Diracology

Using \( \{ \gamma_\mu, \gamma_\nu \} = 2g_{\mu\nu} \) prove
(i) \( \{ \gamma_5, \gamma_\mu \} = 0 \)
(ii) \( \not{p} \not{p} = p^2 \)
(iii) \( Tr(\not{a} \not{b}) = 4a \cdot b. \)
(iv) \( Tr(\text{odd no } \gamma/\text{s}) = 0. \) (Hint, insert \( \gamma_5^2 = 1 \) in the trace.)
(v) \( Tr(\not{a} \not{b} \not{c} \not{d}) = 4(\not{a} \cdot \not{b})(\not{c} \cdot \not{d}) - (\not{a} \cdot \not{c})(\not{b} \cdot \not{d}) + (\not{a} \cdot \not{d})(\not{b} \cdot \not{c}) \) (Hint: use \( \not{a} + \not{b} = 2a \cdot b \) repeatedly.)
(vi) \( Tr(\gamma_5) = 0. \)
(vii) \( Tr(\gamma_5 \not{a} \not{b}) = 0 \)
(viii) \( Tr(\gamma_5 \not{b} \not{a} d) = 4i\epsilon_{\mu\nu\lambda\sigma} a^\mu b^\nu c^\lambda d^\sigma. \)
(ix) \( \gamma_\mu \gamma^\mu = 4 \)
(x) \( \gamma_\mu \not{b} \gamma^\mu = -2 \not{a} \)
(xi) \( \gamma_\mu \not{b} \not{c} \gamma^\mu = 4a \cdot b \)
(xii) \( \gamma_\mu \not{b} \not{c} \not{d} \gamma^\mu = -2 \not{c} \not{b} \not{a} \)

2. conservation of total spin

Show that \( [H, \vec{J}] = 0 \) where \( H = \alpha \cdot \vec{p} + \beta m \) and \( \vec{J} = \vec{L} + \frac{1}{2} \vec{S} \) with \( \vec{S} = diag(\vec{\sigma}, \vec{\sigma}) \).

3. Lorentz invariance

(i) Show \( S^{-1} \gamma^\mu S = \Lambda^\mu_\nu \gamma^\nu \) to first order. Thus \( \gamma^\mu \) transforms as a 4-vector.
(ii) Prove \( S^{-1} = \gamma^0 S \gamma^0 \).
(iii) Prove \( \gamma^5 S = S \gamma^5 \).

4. Parity

(i) Give the explicit transformation properties of all bilinears under parity.
(ii) People often wonder how the reaction \( \pi \rightarrow \mu \nu \) can proceed since the pion has \( J = 0 \) while the \( W \) is a vector (\( J = 1 \)) particle. Explain. Note that the current associated with the \( W \) is proportional to \( V^\mu - A^\mu = \bar{\psi} \gamma^\mu (1 - \gamma_5) \psi \).

5. Bilinears

(i) Prove that the operator \( \bar{\psi} \psi \) is a scalar by establishing the quantum numbers of the state \( \int d^3 x \bar{\psi}(\vec{x}) \psi(\vec{x}) |0\rangle \) in the nonrelativistic limit. Work in the rest frame of the created particle.
(ii) Do the same for the vector current. What happens for \( \int d^3 x \bar{\psi} \gamma^0 \psi |0\rangle \)?
Hint: a fermion-antifermion in the state $^{2S+1}L_J$ has parity $P = (-1)^{L+1}$ and charge conjugation $C = (-1)^{L+S}$. Prove these relations for bonus points.

6. Chirality and Helicity

(i) Show that chirality is not a good quantum number for a massive fermion.

(ii) Show that helicity is conserved for a massive fermion.

(iii) Show that $\gamma_5 u = \Sigma \cdot \hat{p} u$ in the high energy limit where $\Sigma = diag(\sigma, \sigma)$. Thus chirality and helicity are the same in the high energy limit.