4. Spontaneously Broken Field Theory

1. A Zeroth-order Natural Relation (Peskin and Schroeder, 11.2, plus some more)

We consider the $O(2)$ linear sigma model coupled to fermions:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}\mu^2(\phi^i)^2 - \frac{\lambda}{4}[(\phi^i)^2]^2 + \bar{\psi}(i\not{\partial})\psi - g\bar{\psi}(\phi^1 + i\gamma_5\phi^2)\psi. \qquad (1)$$

(a) Show that the theory has the following global symmetry:

$$\begin{aligned} \phi^1 &\rightarrow \cos\alpha\,\phi^1 - \sin\alpha\,\phi^2, \\ \phi^2 &\rightarrow \sin\alpha\,\phi^1 + \cos\alpha\,\phi^2, \\ \psi &\rightarrow \mathrm{e}^{-i\alpha\gamma_5/2}\,\psi. \end{aligned} \qquad (2)$$

Show that the solution to the classical equations of motion with the minimum energy breaks this symmetry.

(b) Let $\phi^1 = v + \sigma(x)$ and $\phi^2 = \pi(x)$ and write out the complete Lagrangian in these new variables. Write out the complete counterterm Lagrangian. You should find 12 terms in $\mathcal{L}$ and 14 terms in $\delta\mathcal{L}$. Show that the fermion acquires a mass $m_f = vg$ in the broken phase of the theory.

(c) Consider the one-loop radiative corrections to $m_f = vg$: choose renormalisation conditions such that v receives no radiative corrections at one-loop and g as defined by the $\bar{f}f\pi$ vertex also gets no radiative corrections at zero pion momentum. Show that m_f is no longer equal to vg (you may not have guessed this!). Also show that m_f is automatically ultraviolet finite at this order.

(d) Obtain explicit expressions for δ_g (the divergent part of the $\bar{f}f\pi$ vertex) and for the finite correction to $m_f = vg$. You will find the discussion on pages 189-194 and Appendix A.4 useful.