7. [QFTII, 2013] Schwinger-Dyson Equations

1. A σ–π Ward Identity (Amit, 5.11)

Derive the following

$$\Gamma_{\pi\pi}(p) - \Gamma_{\sigma\sigma}(p) = -v\Gamma_{\sigma\pi\pi}(p, 0, -p) \tag{1}$$

with the aid of the Ward-Takahashi identity for the effective action. What happens as $p \to 0$?

2. ϕ^4 theory

Derive the Schwinger-Dyson equation for ϕ^4 theory for the full 1PI propagator. Give a diagrammatic version of your analytic result.

3. Yukawa theory

(i) Obtain the Schwinger-Dyson master equation for the Yukawa theory with generating functional

$$Z[J, \eta, \bar{\eta}] = \int D\bar{\psi} D\psi D\sigma \, \mathrm{e}^{i\int d^4x \mathcal{L} + i\int (J\sigma + \bar{\eta}\psi + \bar{\psi}\eta)}$$

with

$$\mathcal{L} = \bar{\psi}(i \not{\partial} - m)\psi - g\bar{\psi}\sigma\psi + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2}m^2\sigma^2.$$

Hint: start with

$$\int D\bar{\psi} D\psi D\sigma \left(\frac{\delta S}{\delta \bar{\psi}} + \eta \right) \mathrm{e}^{\cdots} = 0.$$

(ii) Obtain the Schwinger-Dyson equation for the $\sigma\bar{\psi}\psi$ vertex.

4. Rainbow-ladder QED

Draw the Schwinger-Dyson equation for the electron propagator in QED.

(i) Take the full electron propagator to be of the form

$$S(p) = \frac{i}{A(p^2)\not{p} + B(p^2)},$$

and assume that the full photon-electron vertex is given by its tree order value. Obtain integral equations for A and B. Work in Landau gauge.

(ii) Set ($A = 1$) and concentrate on the equation for B. Assume a momentum cutoff of Λ. Set $m = 0$, Wick rotate, and do the angular integrals analytically.

(iii) Convert the integral equation of (ii) to a differential equation by repeatedly differentiating wrt p_E^2. Obtain the following boundary conditions:

$$(p_E^4 B')|_{p=0} = 0$$

and

$$(B + p_E^2 B')|_{p=\Lambda} = 0$$

.

(iv) Obtain the analytic solution for B at large (Euclidean) momentum and prove that the boundary conditions can only be met if α is larger than some critical coupling. Determine this critical coupling and interpret your results.