1. **A $\sigma-\pi$ Ward Identity** (Amit, 5.11)

Derive the following

$$\Gamma_{\pi\pi}(p) - \Gamma_{\sigma\sigma}(p) = -v\Gamma_{\sigma\pi\pi}(p,0,-p)$$  \hspace{1cm} (1)

with the aid of the Ward-Takahashi identity for the effective action. What happens as $p \to 0$?

2. **A Zeroth-order Natural Relation**

We consider the $O(2)$ linear sigma model coupled to fermions:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}\mu^2(\phi^i)^2 - \frac{\lambda}{4}[(\phi^i)^2]^2 + \bar{\psi}(i \not\partial)\psi - g\bar{\psi}(\phi^1 + i\gamma^5\phi^2)\psi.$$  \hspace{1cm} (2)

(a) Show that the theory has the following global symmetry:

$$\phi^1 \to \cos \alpha \phi^1 - \sin \alpha \phi^2,$$
$$\phi^2 \to \sin \alpha \phi^1 + \cos \alpha \phi^2,$$
$$\psi \to e^{-i\alpha\gamma_5/2}\psi.$$  \hspace{1cm} (3)

Show that the solution to the classical equations of motion with the minimum energy breaks this symmetry.

(b) Let $\phi^1 = v + \sigma(x)$ and $\phi^2 = \pi(x)$ and write out the complete Lagrangian in these new variables. Write out the complete counterterm Lagrangian. You should find 12 terms in $\mathcal{L}$ and 14 terms in $\delta\mathcal{L}$. Show that the fermion acquires a mass $m_f = vg$ in the broken phase of the theory.

(c) Consider the one-loop radiative corrections to $m_f = vg$: choose renormalisation conditions such that $v$ receives no radiative corrections at one-loop and $g$ as defined by the $\bar{f}f\pi$ vertex also gets no radiative corrections at zero pion momentum. Show that $m_f$ is no longer equal to $vg$ (you may not have guessed this!). Also show that $m_f$ is automatically ultraviolet finite at this order.

(d) Obtain explicit expressions for $\delta g$ (the divergent part of the $\bar{f}f\pi$ vertex) and for the finite correction to $m_f = vg$. You will find the discussion on pages 189-194 and Appendix A.4 useful.