5. Young Tableaux

Young tableaux are graphical representations of irreps that correspond to tensors. Tableaux (the singular is tableau) are drawn as connected boxes. For example, the fundamental rep is drawn as \[ \begin{array}{c} \hline \end{array} \] A rank \( n \) symmetric tensor is written as a row of \( n \) boxes: \[ S^{ijk} = \begin{array}{ccc} & & \end{array} \] whereas a rank \( n \) antisymmetric tensor is a column of \( n \) boxes, eg, \[ A^{ij} = \begin{array}{cc} \hline \end{array} \] A general tableau will be of mixed symmetry. A tableau can consist of any number of boxes provided that no more than \( N \) boxes occur in any column (for SU(N)), the rows start at the left, and no row is longer than a row above it. Thus the following are valid tableaux

\[ \begin{array}{ccc} & & \end{array} \]

whereas the following are not

\[ \begin{array}{ccc} & & \end{array} \]

The dimension of an SU(N) irrep is given by the formula

\[ D = \prod_{\text{boxes}} \frac{(N + d_{\text{box}})}{h_{\text{box}}} \]

where \( d_{\text{box}} \) is assigned as shown here:

\[ \begin{array}{cccc} 0 & 1 & 2 & 3 \\
-1 & 0 & 1 & 2 \\
-2 & -1 & 0 & 1 \\
\end{array} \]

etc. The \( h_{\text{box}} \) are called the hook lengths and are given by one plus the number of boxes to the right of the box plus the number of boxes below the box. For example, hook lengths for \[ \begin{array}{cc} \hline \end{array} \] are given by

\[ \begin{array}{c} 4 \ 2 \ 1 \ 1 \\
\ 1 \end{array} \]

Thus, in SU(N) we have the following dimensions (for example):

\[ \begin{array}{c} \end{array} = N \]
\[ \begin{array}{ccc} & & \end{array} = \frac{1}{3} N(N+1)(N-1) \]
\[ \begin{array}{cccc} & & & \end{array} = \frac{1}{80} N^2(N^2-1)(N+2)(N+3) \]

Finally, tableaux can be combined to determine irreps of products of irreps, say \( T_1 \otimes T_2 \). To do this

(i) assign distinct labels to boxes in each row of \( T_1 \) as follows, eg, \[ \begin{array}{ccc} & & \end{array} \]

(ii) attach boxes labelled by \( a \) to \( T_2 \) in all possible ways such that no two \( a \)s appear in the same column and the result is still a Young tableau.
(iii) Repeat with BS, CS, etc.

(iv) After all the boxes of $T_1$ have been added to $T_2$ form the sequence for each tableau by reading each row from right to left while going from top to bottom. Eliminate tableaux with inadmissible sequences, namely, when reading the sequence (from left to right) there must be at least as many $a$s as $b$s, $b$s as $c$s, etc at any point in the sequence. Any resulting tableau with the same structure and sequence are copies of each other and only one tableau should be considered.

Some examples:

$$N \otimes \frac{1}{2} N(N+1) = \begin{array}{c} \text{3 boxes} \end{array} = \begin{array}{c} \text{3 boxes} \end{array} = \begin{array}{c} \text{3 boxes} \end{array} = \frac{1}{6} N(N+1)(N+2) + \frac{1}{3} N(N^2-1).$$

The sequences for these are $baa$, $aab$, $aba$, $aab$, $baa$, $aba$, $aba$, $aab$. All are admissible except $baa$ so we remove tableaux 1, 4, 7, and 9. The remaining have the following dimensions in SU(3): 27, 10, 10 (a different 10!), 8 (different from $\mathbf{3}$), 1, 8 (notice another way to write the singlet), null. Thus

$$8 \otimes 8 = 1 \oplus 8' \oplus 8' \oplus 10 \oplus 10' \oplus 27.$$