Derivatives for Phys0104

Definition:

$$\frac{df}{dx} = f' = \frac{d}{dx} f = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

- If $f(x) = x$ then $f'(x) = \lim_{\Delta x \to 0} \frac{(x+\Delta x) - x}{\Delta x} = 1$.

- If $f(x) = x^2$ then $f'(x) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0}(2x + \Delta x) = 2x$.

- Use the binomial theorem to prove that $\frac{d}{dx} x^n = n x^{n-1}$.

Pf: By definition the LHS is $[(x+\Delta x)^n - x^n]/\Delta x$. But $(x+\Delta x)^n$ is $\sum_{i=0}^{n} \binom{n}{i} x^i (\Delta x)^{n-i}$. The important terms in this sum are $\binom{n}{n} x^n = x^n$ and $\binom{n}{n-1} x^{n-1} \Delta x$. The first cancels against the $-x^n$ leaving $n x^{n-1} \Delta x$. All other terms have multiples of Δx^2 or higher so they vanish when we divide by Δx and take $\Delta x \to 0$. This proves the theorem.

- Prove $\frac{d}{dx}(K f(x)) = K \frac{df(x)}{dx}$.

Pf: By definition the LHS is $[K f(x+\Delta x) - K f(x)]/\Delta x$, but this is just $K[f(x+\Delta x) - f(x)]/\Delta x = K f'(x)$.

- Prove $\frac{d}{dx}[f(x) + g(x)] = \frac{df}{dx} + \frac{dg}{dx}$.

Pf: Again, use the definition on the LHS. We have

$$\frac{[f(x+\Delta x) + g(x+\Delta x)] - [f(x) + g(x)]}{\Delta x} = \frac{f(x+\Delta x) - f(x) + g(x+\Delta x) - g(x)}{\Delta x} =$$

$$f' + g'.$$

- Product Rule: If $h(x) = g(x) f(x)$ then $h'(x) = g(x) f'(x) + g'(x) f(x)$.

Pf: By definition

$$h'(x) = \lim_{\Delta x \to 0} \frac{g(x+\Delta x) f(x+\Delta x) - g(x) f(x)}{\Delta x}.$$

Now invert the definition of the derivative to write $f(x+\Delta x) \approx f(x) + \Delta x f'(x)$. Substitute this expression for $f(x+\Delta x)$ and $g(x+\Delta x)$ to get

$$h'(x) = \lim_{\Delta x \to 0} \frac{(g(x) + \Delta x g'(x))(f(x) + \Delta x g'(x)) - g(x) f(x)}{\Delta x} =$$

$$\lim_{\Delta x \to 0} [g(x) f'(x) + g'(x) f(x) + \Delta x f'(x) g'(x)] = g(x) f'(x) + g'(x) f(x).$$

- Example. Let $f = x^2$, $g = x^3$. Then $h = x^5$. So we can check our expressions: $h' = 5x^4$ and $f'g + fg' = 2x \cdot x^3 + x^2 \cdot 3x^2 = 5x^4$. $\surd$

• Chain Rule: if $h(x) = f(g(x))$ then $h'(x) = \frac{df}{dg}\frac{dg}{dx}$.
Pf:

$$h'(x) = \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(g(x) + \Delta x g'(x) + \ldots) - f(g(x))}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(g(x)) + \Delta x g'(x)\frac{df}{dg} - f(g(x))}{\Delta x} = \frac{df}{dg} g'(x) = \frac{df}{dg}\frac{dg}{dx}.$$

• Example: consider $h(x) = (\sin(x))^2$. We can think of this as $h = f(g)$ with $f(g) = g^2$ and $g(x) = sin(x)$. Then the chain rule gives

$$h'(x) = \frac{df}{dg}\frac{dg}{dx} = \frac{d}{dg}(g^2)\frac{d}{dx}\sin(x) = 2g\cos(x) = 2\sin(x)\cos(x).$$

• Example: consider $h(x) = \sin(x^2)$. Now we take $f = \sin(g)$ and $g = x^2$. Then

$$h'(x) = \frac{df}{dg}\frac{dg}{dx} = \frac{d}{dg}(\sin(g))\frac{d}{dx}x^2 = \cos(g)2x = \cos(x^2) \cdot 2x.$$

You can think of this as "take the derivative of the outside and multiply it by the derivative of the inside".

Some other derivatives you'll need to know for Phys0104:

$$\frac{d}{dx}\sin(x) = \cos(x) \qquad\qquad \frac{d}{dx}\cos(x) = -\sin(x)$$